Section 4. Natural Sciences and Informational Tehnology

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Notice on Relative Weak Extension Property

A *quasivariety* of algebras is an universal Horn class of algebras containing the one element algebra. Equivalently, a quasivariety is a class of algebras of the same similarity type which is closed under isomorphic images, subalgebras, direct products (including the direct product of an empty family) and ultraproducts. A *variety* is a quasivariety which is closed under homomorphic images. Further we assume that all considered algebras are of the same finite similarity type and all the classes of algebras to be abstract, that is, together with any algebra, such class contains all its isomorphic copies.

Let *R* be a quasivariety. A congruence θ on algebra A is called an *R-congruence* provided A/θ ∈ *R*. The set Con_RA of all *R*-congruences of A forms a complete algebraic lattice which is a meet-subsemilattice of the lattice *Con* A of all congruences of A. These lattices have the same the least and largest elements, denoted 0_A and 1_A. They are the identity relation and the universal relation over A. For each subset X of AЧA there exists the least *R*-congruence containing X and we denote it by Cg_R(X). If X consists of one element {(a,b)} we simply write Cg_R(a,b) and call such a congruence an *R-principal congruence* or a *relative principal congruence*. They are the identity relation and the universal relation over A. In event *R* is a variety, the lattice Con_R A coincides with *Con* A.

Let *R* be a subquasivariety of quasivariety *K*. A quasivariety *R* is said to have the *weak extension property relative K* (*R* |= *K*-WEP, in short) if for all A ∈ *R* and α, β ∈ Con_K A, $\alpha \cap \beta = 0_A$ implies $Cg_R(\alpha) \cap Cg_R(\beta) = 0_A$. If *K* is clear under the context we say that *R* has relative weak extension property and write *R* |= RWEP. If *K* is a variety then the definition of RWEP coincides with usual definition of weak extension property.

The concept of the weak extension property was introduced by K.Kearnes and R.McKenzie in [4], where they extended Commutator Theory into relative congruence modular quasivariety and, in particular, proved that relative congruence modular quasivariety has the weak extension property. Also every relative semi-distributive quasivariety has the weak extension property. This result was obtained in [2].

It is easy to see that every quasivariety *R* having WEP has a weak extension property relative any quasivariety *K* containing *R*. Also every relative subvariety of given quasivariety *K* has *K*-WEP. In this notice we show that there exist quasivarieties *R* and *K* such that *R* has *K*-WEP, and *R* is not relative variety and has no WEP. Also we extend some basic results for quasivarieties with WEP on quasivarieties with RWEP.

The basic results and definitions on Universal algebra and Lattice theory can be found in textbooks[1,3,5].

1. Quasivarieties with RWEP. The natural examples of quasivarieties with RWEP are:

i) *R* is relative subvariety of *K*, that is, *R* is an intersection of quasivariety *K* and some variety *V*.

ii) If R has WEP then R has *K*-WEP for every quasivariety *K* containing *R*.

The main purpose of this section is to show that there are quasivarieties *R*, *K* such that *R* has *K*-WEP, *R* is not relative subvariety of *K* and *R* has no WEP.

For this we need the following necessary and sufficient conditions of RWEP for locally finite quasivarieties.

Lemma 1. Let *R* be a subquasivariety of locally finite quasivariety *K*. Then *R* has the weak extension property relative *K* if and only if for all finite $A \in R$ and $\alpha \in Con_K A$, if β is maximal in $Con_K A$ with respect to satisfying $\alpha \cap \beta = 0_A$, then $\beta \in Con_R A$.

Proof. $\Rightarrow$. Let $A \in R$ be a finite algebra, $\alpha \in Con_K A$ and $\alpha \cap \beta = 0_A$ for some congruence β maximal in *Con* A. By *K*-WEP, we get $Cg_R(\alpha) \cap Cg_R(\beta) = 0_A$. Since β maximal in *Con* A we obtain $\beta = Cg_R(\beta)$. That is, β is an *R*-congruence.

$\Leftarrow$. Let $\alpha \cap \beta = 0_A$ for some *K*-congruences α, β on A. Since A is a finite there is a maximal *K*-congruence α' such that $\alpha \subseteq \alpha'$ and $\alpha' \cap \beta = 0_A$. By condition of lemma, $\alpha' \in Con_R A$. By the same arguments, there is *R*-congruences β' such that $\beta \subseteq \beta'$ and $\alpha' \cap \beta' = 0_A$. So as $Cg_R(\alpha) \subseteq \alpha'$ and $Cg_R(\beta) \subseteq \beta'$ we obtain $Cg_R(\alpha) \cap Cg_R(\beta)=0_A$.

The similar condition for WEP was found in [6].

<u>Example</u>. Let A be the five elements meet semilattice with the least and the largest elements 0 and 1, respectively, and additional one unary operation f, four constants a, b, c, 0 satisfying the following relations $a < c$, $b < c$, $b \neq a$, $f1 = 0$ and $fx = x$ for all $x \in \{a, b, c, 0\}$. Let *R* be the least quasivariety containing algebra A, and *K* the least quasivariety containing algebras A, A_c where $A_c = A/\theta(0,c)$. Then *R* has *K*-WEP and, also *R* is not relative subvariety of *K* and has no WEP.

Proof. Since the quasi-identity

$$(\forall xy)[c = 0 \rightarrow x = y],$$

is true in A and false in A_c we get *R* is not relative subvariety of *K*.

Since the quasi-identities

$$(\forall xy)[a = 0 \rightarrow x = y] \text{ and } (\forall xy)[b = 0 \rightarrow x = y]$$

are true in A we get $\theta_R(0, a) = \theta_R(0, b)$. On the other hand, $\theta(0, a)\cap\theta(0, b) = 0_A$. Therefore, *R* has no WEP.

Let A_0 be the four elements subalgebra of A. Since A has one proper subalgebra A_0 the class of all relative subdirectly *R*-irreducible algebras consists A and A_0. By the same argument, the class of all relative subdirectly *K*-irreducible algebras consists A, A_0 and A_c.

Now let $B \in R$ be a finite subdirect product of algebras A_i , $i = 1,..., k$, and $A_i \in S(A)$ for all $i = 1,..., k$. Let $\theta_K(u, v)$ be a minimal congruence in Con_K B for some $u,v \in B$. Since $(u \wedge w,v \wedge w) \in \theta_K(u, v)$ for any $w \in B$ and $\theta_K(u, v)$ is minimal we may assume $u > v$ and u, v are minimal elements with such property. Put $u = (u_1,..., u_k)$, $v = (v_1,..., v_k)$ and $v_i < u_i$ for some $i \leq k$. Assume $u_i=1$. Then $v_i < 1$ and, by definition of operation f, we have $f(1) = 0$ and $f(v_i) = v_i$. Hence $f(u) \leq u$, $f(v) \leq v$ and f(v) is not less then f(u), or $f(u) = f(v)$.

Suppose $v_i < c$. Since element $c = (c ,..., c)$ belongs to B we have $v\wedge c < u\wedge c < u$ and $(v\wedge c, u\wedge c) \in \theta(u,v)$. This is impossible because pair (u,v) is minimal. Hence $v_i = c$. Since $(f(u),f(v)) \in \theta_K(u,v)$ we have $(f(u) \wedge f(v),f(v)) \in \theta_K(u,v)$ and $f(u) \wedge f(v) < u$. This is impossible by minimality of pair (u,v). So we have $u_i < 1$ for all $i = 1,..., k$. Therefore, we obtain $(0,..., 0) \leq v < u < (c,..., c)$.

Let γ be a maximal *K*-congruence such that $\gamma \cap \theta_K(u,v) = 0$. Then the factor-algebra B/γ is relative subdirectly *K*-irreducible algebra, that is, $B/\gamma \in \{A, A_0, Ac\}$. If $B/\gamma \in \{A, A_0\}$ then $\gamma \in Con_K$ B. Now, let $B/\gamma = A_c$. Then $((0,..., 0), (c,..., c)) \in \gamma$. Since $(0, ...,0) < v < u < (c,..., c)$ we have $(u,v) \in \gamma$. But $\gamma \cap \theta_K(u,v) = 0$. Contradiction. Hence $\gamma \in Con_K$ B. By Lemma 1, we get that *R* has weak extension property relative *K*.

2. Covers of quasivarieties with RWEP. The set of all subquasivarieties of given quasivariety *R* forms a complete coalgebraic lattice under inclusion which is called quasivariety lattice of *R*. In [2] have been proved that if *R* is a quasivariety with the weak extension property that is included in a finitely generated quasivariety *K*, then *R* is finitely axiomatizable relative to *K*. Actually, it was proved that *R* has a finite number covers in the quasivariety lattices of *K*. The idea of proof the above result gives us possibility to extend this theorem to the quasivarieties having relative weak extension property.

Theorem. Let *R, K* are quasivarieties, $R \subseteq K$ and *K* a finitely generated quasivariety. If *R* has *K*-WEP, then *R* has a finite number covers in the quasivariety lattice of *K*. In particular, if *K* is finitely axiomatizable then *R* is finitely axiomatizable too.

Proof. We just repeat arguments of the proof of Theorem 4 [2] with some additions.

Let n be an integer not less than the cardinality of any of the generators of *K*. Let m be the maximum size of an n-generated subalgebra of any algebra in *K*. We may assume that $R \subset K$. Assume A is a finite algebra in $K \setminus R$. For $X, Y \subseteq Con_K A$, we define X « _Y iff for every $\_\alpha \in Y$ there exists $\_\beta \in X$ such that $\_\beta \leq \alpha$. Notice that « restricted to antichains in Con_K A is a partial order. Let X be an antichain in Con_K A whose intersection is 0_A and such that for each $\_\alpha \in X$, the quotient algebra $A/\alpha\_$ has at most n elements. Notice that such X exists for $A \in K$ since n bounds the cardinality of the generators of *K*. Now, among all antichains Y in Con_K A with X « Y and with the intersection of Y equal to 0_A, we choose one, say Z, that is _«-maximal. It follows from the choice of Z that each member of Z is a meet-irreducible element of Con_K A. Since A is outside of *R*, we can choose γ in Z so that $A/\gamma \notin R$. Let $\delta\_$ be the unique cover of γ in Con_K A. Since, by the choice, Z is «-maximal, there is a pair (a, b) in AxA such that $(a, b) \in \delta \cap \cap(Z\setminus\{\gamma\})$ and $a \neq b$. Choose $a_0, \ldots, a_{k-1}$ in A to be a selector set for all the γ – equivalence classes in A with $a_0 = a$ and $a_1 = b$. As X « Z, it follows that $k \leq n$.

Let B be the subalgebra of A generated by $a_0, \ldots, a_{k-1}$. As $k \leq n$, B has at most m elements. Let γ' and σ denote the restrictions of γ and $\cap (Z \setminus \{\gamma\})$ to the algebra B, respectively. Obviously, $\gamma' \cap \sigma = 0_B$. We now want to show that $B \notin R$. To this effect suppose otherwise that $B \in R$. Then, by *K*-WEP of *R*, there are *R*-congruences γ'_R and σ_R on B that extend γ' and σ, respectively, and are such that $\gamma'_R \cap \sigma_R = 0_B$. As $a, b \in B$ and $(a, b) \in \cap(Z \setminus \{\gamma\})$, it follows that $(a, b) \in \sigma\_ \leq \sigma_R$, This implies (a, b) $\notin \gamma'_R$. As B is a subalgebra of A that is generated by a selector set for all the γ-equivalence classes in A, it follows that B/γ' is isomorphic to A/γ. Thus γ' has the unique cover in Con_K B to which (a, b) belongs. This implies that $\gamma'_R = \gamma'$ for otherwise γ'_R would be above this unique cover and as a consequence (a, b) would belong to γ'_R. But A/$\gamma \notin R$ and so B/$\gamma' \notin R$. Hence B/$\gamma'_R \notin R$, a contradiction. Thus B $\notin R$. _

So we obtain that any finite algebra in $K \setminus R$ has a subalgebra of at most m elements that lies outside of R. Since K is a locally finite, the number of m-elements algebras is finite. This means that R has a finite number covers in the quasivariety lattice of K.

To prove the second part of theorem, it is enough to show that R has a finite quasi-equational basis relative K.

The fact that any finite algebra in $K \setminus R$ has a subalgebra of at most m elements which lies outside of R implies that there is no decreasing chain of quasivarieties reaching on the quasivariety R. Let $R_1, R_2, ..., R_k$ are all covers of quasivariety R. And let $q(i)$ be a quasi-identity that is true in R and false in R_i, i = 1,2,...,k. Put S=Mod($\{q(i) : i<k+1\}$) $\cap K$. It is easy $R \subseteq S$. Suppose $R \neq S$. Since there are no decreasing chains reaching on R we get S contains R_i for some $i \leq k$. Therefore, $q(i)$ is false in R_i. This is impossible by definition of $q(i)$. Thus R=S.

In the proof of theorem we use the fact that every n-generated algebra from R has K-WEP. So we get

Corollary. Let R, K are quasivarieties, $R \subseteq K$ and K residually less then n. If any n-generated algebra in R has K-WEP, then R has a finite number covers in the quasivariety lattice of K. In particular, R is finitely axiomatizable relative to K.

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