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MATHEMATICS

FOR ECONOMICS, BUSINESS AND SOCIAL SCIENCES

Linear Algebra and Analytic Geometry

Dear readers!
It is our pleasure to introduce this textbook to you.

For many of you, this textbook is one of the first books that will help you gain knowledge within the walls of AUCA – one of the best universities in Central Asia.

We hope that your future successes in science, business, and politics will contribute to expanding the borders of the region, in which AUCA will be called one of the best, thanks to you.

One of the famous contemporary writers, Yuval Noah Harari in his book **Homo Deus: A Brief History of Tomorrow** wrote: “In medieval Europe, the chief formula for knowledge was:

$$\textit{Knowledge} = \textit{Scriptures} \times \textit{Logic}.$$

If we want to know the answer to some important question, we should read scriptures, and use our logic to understand the exact meaning of the text.

The Scientific Revolution proposed a very different formula for knowledge:

$$\textit{Knowledge} = \textit{Empirical Data} \times \textit{Mathematics}.$$

If we want to know the answer to some question, we need to gather relevant empirical data, and then use mathematical tools to analyze the data”

In other words, Mathematics is a universal tool for the study of social, natural sciences, and other problems.

We wish you success in the study and creative mastery of not only Mathematics but all other disciplines both within the walls of AUCA and throughout your life.

Contents

§1. Introduction to systems of linear algebraic equations

When you are studying mathematics, especially if you are not going to become professional mathematicians, you always need to ask the question: “Where can this be applied?” The answer will be very simple if we start studying Linear Algebra and Analytic Geometry with systems of linear algebraic equations.

1.1. Degenerate systems

Problem

Anara bought 17 notebooks for 306 soms. Erkingul bought 13 notebooks and 19 pens for 671 soms. Determine the price of the notebook and the price of the pen.

Solution

We denote the price of the notebook by n, the price of the pen by p, then we get the system $\begin{cases} 17n = 306, \\ 13n + 19p = 671. \end{cases}$

Such systems are called degenerate systems and finding their solution is quite simple:

First, you need to solve an equation containing only one unknown, and using the value found, we can find another. So, from the first equation: $n = 306/17 = 18$. Then $13 \cdot 18 + 19p = 671$.

Hence, $19p = 671 - 234$ and $p = 437/19 = 23$.

So, we found that the price of the notebook is 18 soms and the price of the pen is 23 soms.

Exercises 1.1

C. Arstan bought 7 brushes and 11 albums for 732 soms. Chinara bought 15 albums for 645 soms. Determine the price of the brush and the price of the album.

H. Elena bought 9 paint cans for 657 soms. Sergey bought 5 paint cans and 29 rolls of wallpaper for 6542 soms. Determine the price of a paint can and the price of a roll of wallpaper.

1.2. Substitution method

Problem

Aizhan bought 17 jumpsuits and 8 shirts for 5310 soms. The price of the jumpsuit was 55 soms more than the price of the shirt. How many soms did Gulnura pay for 7 jumpsuits and 3 shirts in the same store?

Solution

We denote the price of the jumpsuit by *n*, the price of the shirt by *s*, we get the system $\begin{cases} 17n + 8s = 5310, \\ n - s = 55. \end{cases}$ The value of *n* can easily be expressed from the 2nd equation of the system, it is convenient to solve it by the substitution method: $\begin{cases} 17n + 8s = 5310, \\ n = s + 55, \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 17(s + 55) + 8s = 5310, \\ n = s + 55, \end{cases} \Leftrightarrow \begin{cases} 25s = 5310 - 935, \\ n = s + 55. \end{cases}$$

The result is a degenerate system.

Therefore, from the first equation: $s = 4375/25 = 175$.

Then $n = s + 55 \Rightarrow n = 175 + 55 = 230$.

We found that for 7 jumpsuits and 3 shirts Gulnura paid: $7n + 3s = 7 \cdot 230 + 3 \cdot 175 = 2135$ soms.

Exercises 1.2

C. Aybike got 71 “fives” and “fours”. Moreover, the number of “fours” is three times less than the number of “fives” with one more. Determine the number of “fives” and the number of “fours”.

H. The perimeter of the rectangle is 97 centimeters. The length is 5 cm more than the double width. Determine the area of this rectangle.

1.3. "Addition–multiplication" method 1

Problem

The total weight of 7 bulls and 23 sheeps in Koshoy's herd is 2425 kilograms. After he sold 10 sheeps and bought another 7 bulls, the total weight of his herd became 3299 kg. What is the average weight of a bull and the average weight of a sheep in Koshoy's herd?

Solution

Denote the average weight of the bull by b and the average weight of the sheep by s, then we get the system

$$\begin{cases} 7b + 23s = 2425, \\ 14b + 13s = 3299. \end{cases}$$

Of course, this system can be solved by the substitution method, but the corresponding expressions will be rather cumbersome and inconvenient for calculations. At the same time, many inconveniences can be avoided if you use the method of "addition–multiplication" or as it is called the Gauss method.

In this case, the following property is used for systems:

Solutions of the system will not change if its equation is replaced by the sum of this equation with another equation multiplied by any number.

In this case, we notice that the coefficients of ***b*** are “close relatives”. Therefore, we multiply the first equation of the system by (-2) and add to the second:

$$\begin{cases} 7b + 23s = 2425, \\ -33s = -1551. \end{cases}$$

The result is a degenerate system of equations.

Then: $s = 1551/33 = 47$, and:

$7b + 23\cdot 47 = 2425 \Leftrightarrow b = 1344/7 = 192.$

Thus, the average weight of the bull in the herd is 192 kg, and the average weight of the sheep is 47 kg.

Exercises 1.3

C. The sum of heights of 8 girls and 9 boys in the group is 2755 centimeters. After 4 girls left this group and two boys were added, the sum of heights in the group become 2489 cm. What is the average height of a girl and the average height of a boy in this group?

H. Aizada bought 27 notebooks and 8 pens for 232 soms. In the same store, Gulmira bought 9 notebooks and 14 pens for 253 soms. How many soms did Elnura pay for 11 notebooks and 3 pens in the same store?

1.4. "Addition–multiplication" method 2

Problem

Let us know that 2 tons of flour were transported in a truck in 20 small and 25 large bags. It is impossible to find out the weight of each bag from this information, since this problem has many solutions. For example, if the weight of a small bag is 10 kg, then the weight of a large bag

$(2000 - 20 \cdot 10):25 = 72$ *kg*; if the weight of a small bag is 30 kg, then the weight of a large bag

$(2000 - 20 \cdot 30):25=56$ kg, etc.

This usually happens when the number of unknowns (the weight of the large bag and the weight of the small bag) is greater than the number of equations.

Therefore, in order to get an exact answer, the additional information is needed. This can be information about the weight of a bag, about another transportation, etc.

Suppose that it became known that the second truck transported 3 tons of flour in 38 small and 35 large bags. Using this information, we can obtain a system of equations, solve it, and give the answer: the weight of a small bag is 20 kg, and a large one is 64 kg.

If we have information about the third, fourth, etc. trucks, it can be used to check. Let it be known that the third truck transported 2.5 tons of flour in 10 small and 40 large bags.
Substituting data on the weight of the bags:
$10 \cdot 20 + 40 \cdot 64 = 2760$, we get the mismatch. This means that there is an error in the information we received.
Let us consider in more detail the solution to the problem with two trucks. Denote by x the weight of the small bag, y is the weight of the large bag and obtain a system of two equations: $\begin{cases} 20x + 25y = 2000, \\ 38x + 35y = 3000. \end{cases}$
Multiplying the first equation by 38, and the second by 20, we get $\begin{cases} 760x + 950y = 76000, \\ 760x + 700y = 60000. \end{cases}$ As a result, we equated the coefficients of x in the equations of the system. Now subtract the second equation from the first: $250y = 16000$.
Then, $y = 64$.
Substituting the found value in the first equation of the original system, we obtain $20x + 25 \cdot 64 = 2000$.
Then, $x = 20$.
Substituting the found values of $x = 20$ and $y = 64$ into the second equation of the system, we can make sure that the solution is correct.

Exercises 1.4

C. In the store, the price of 13 caps and 18 hats is 5832 soms, and the price of 22 caps and 31 hats is 9995 soms. What is the price of a cap?

H. The company produces drinks Alpha and Beta. The production of 1 liter of Alpha drink requires 12 minutes of special equipment and 5 g of concentrate. For a Beta drink, it takes 18 minutes and 8 g, respectively. It is known that the company used 90 hours of special equipment and 2.32 kg of concentrate. How many liters of each drink were produced?

1.5. Intro to Cramer's rule 1

The system solution method used in the previous paragraph can be generalized. Let's do it.

Consider a system of two linear equations in two variables in a general form:

$$\begin{cases} a_{11}x + a_{12}y = b_1, \\ a_{21}x + a_{22}y = b_2. \end{cases} \qquad (1.1)$$

In order to solve system (1.1), we multiply the first equation by a_{21}, the second by a_{11}, equating the coefficients at x:

$$\begin{cases} a_{21}a_{11}x + a_{21}a_{12}y = a_{21}b_1, \\ a_{11}a_{21}x + a_{11}a_{22}y = a_{11}b_2. \end{cases}$$

Subtract the first equation from the second:

$(a_{11}a_{22} - a_{21}a_{12})y = a_{11}b_2 - a_{21}b_1$. Then,

$$y = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}. \qquad (1.2)$$

In order to find x_1, we do the same: multiply the first equation by a_{22}, the second by a_{12}, equating the coefficients for y:

$$\begin{cases} a_{22}a_{11}x + a_{22}a_{12}y = a_{22}b_1, \\ a_{12}a_{21}x + a_{12}a_{22}y = a_{12}b_2. \end{cases}$$

Subtract the second equation from the first:

$(a_{11}a_{22} - a_{21}a_{12})x = a_{22}b_1 - a_{12}b_2$. Then,

$$x = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{21}a_{12}}. \qquad (1.3)$$

Formulas (2), (3) are valid if $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

Pay attention to the fact that the expressions in formulas (1.2), (1.3) have the same denominator composed of the coefficients of system (1.1). This number is called the determinant of the coefficient matrix of system (1.1):

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and denoted $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

Sometimes the word determinant and *det* are used:

det A is determinant of matrix *A*.

Note that the expressions $a_{11}b_2 - a_{21}b_1$, $a_{22}b_1 - a_{12}b_2$ from the formulas (1.2), (1.3) are the determinants of the matrices $\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}$, $\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}$ respectively.

We got a special case of a rule called the Cramer's rule:
If $\Delta \neq 0$, the solution of system (1.1) can be find by formulas:

$$x = \Delta_x / \Delta; \qquad y = \Delta_y / \Delta. \qquad (1.4)$$

Here Δ_x and Δ_y are the determinants of the matrix obtained from the matrix of coefficients of system (1.1) by replacing the x- or y-column (column of coefficients x or y) with the column of free terms.

For example, for the previous problem of two trucks

$$\Delta = \begin{vmatrix} 20 & 25 \\ 38 & 35 \end{vmatrix} = 20 \cdot 35 - 38 \cdot 25 = -250;$$

$$\Delta_x = \begin{vmatrix} 2000 & 25 \\ 3000 & 35 \end{vmatrix} = 7000 - 75000 = -5000;$$

$$\Delta_y = \begin{vmatrix} 20 & 2000 \\ 38 & 3000 \end{vmatrix} = 6000 - 76000 = -16000.$$

Then, from formulas (1.4): $x = -5000/(-250) = 20$;
$y = -16000/(-250) = 64$.

Exercises 1.5

Solve the problems from exercises 1.4 using the Cramer's rule.

C. In the store, the price of 13 caps and 18 hats is 5832 soms, and the price of 22 caps and 31 hats is 9995 soms. What is the price of a cap?

H. The company produces drinks Alpha and Beta. The production of 1 liter of Alpha drink requires 12 minutes of special equipment and 5 g of concentrate. For a Beta drink, it takes 18 minutes and 8 g, respectively. It is known that the company used 90 hours of special equipment and 2.32 kg of concentrate. How many liters of each drink were produced?

1.6. Intro to Cramer's rule 2

In order to show that the use of the "substitution" or "multiplication–addition" method is not always rational, we propose to consider the following

Problem

Saadat sells sweets and cookies. On the first day, she sold 17 kilograms of sweets and 22.5 kg of cookies. On the second day, she sold 25.2 kg of sweets and 19 kg of cookies. Determine the prices of sweets and cookies, knowing that the revenue on the first day was equal to 9210 soms, on the second day, the revenue was equal to 10868 soms.

Solution

It is easy to see that the answer can be obtained by solving the system

$$\begin{cases} 17a + 22.5b = 9210, \\ 25.2a + 19b = 10868. \end{cases}$$

Of course, this system can be solved in the way that you know best: express one of the variables from the first equation of the system and substituting into the second, get a linear equation with one unknown. You will have to deal with rather cumbersome transformations.

At the same time, with a calculator, this system is easily solved by the Cramer's rule. So, we calculate the determinant of the matrix of coefficients of the system:

$$D = \begin{vmatrix} 17 & 22.5 \\ 25.2 & 19 \end{vmatrix} = 323 - 567 = -244.$$

(For convenience, instead of the Greek letter Δ, can be used the Latin letter D.)

Next, we calculate the determinants given by the right-hand side of the system:

$$D_a = \begin{vmatrix} 9210 & 22.5 \\ 10868 & 19 \end{vmatrix} = 174990 - 244530 = -69540;$$

$$D_b = \begin{vmatrix} 17 & 9210 \\ 25.2 & 10868 \end{vmatrix} = 184756 - 232092 = -47336.$$

Then, from Cramer's formulas: $D \cdot a = D_a$; $D \cdot b = D_b$ obtain that

$a = (–69540)/(–244) = 285$; $b = (–47336)/(–244) = 194$.

Thus, it was found out that a kilogram of sweets cost 285 soms, a kilogram of cookies cost 194 soms.

Exercises 1.6

C. Olga bought 5 kilograms of meat and 8 kg of potatoes for 1820 soms. If meat were 20% more expensive, and potatoes were 10% cheaper, then it would have spent 2 148 soms. What is the price of meat?

H. The company produces Alpha and Beta TVs from the available parts. For the production of 1 Alpha TV, 4.8 hours for assembly and 2.7 hours for setup are used. For the production of 1 Beta TV, 6.1 hours in assembly and 3.3 hours in setup are used. How many TVs of each brand were manufactured if 6 949 hours were spent on assembly and 3 813 hours spent on setup?

1.7. Other method 1

The substitution method, the Gauss method, and the Cramer rule are most often used in solving systems of linear algebraic equations. However, in some cases, other approach can be used. Therefore, the general recommendation — before you start to solve the problem, think a little.

Problem

Marina chose four books in the bookstore. However, going to the cash desk, she found that there was only enough money for three books. Moreover, if she refuses the first book, then she will have to pay 927 soms, from the second — 904 soms, from the third — 931 soms, from the fourth — 898 soms. How much is each book?

Solution

Denoting the price of the first book by *f*, the second by *s*, the third by *t*, the fourth by *r*,
we get a system of four equations with four unknowns:

$$\begin{cases} s + t + r = 927; \\ f + t + r = 904; \\ f + s + r = 931; \\ f + s + t = 898. \end{cases}$$

It looks, of course, terrifying — four equations, four unknowns.
But, if you simply add up these equations, you get the equation $3(f + s + t + r) = 3660$.
Hence, $f + s + t + r = 1220$.

Now, sequentially subtracting the first, then the second, then the third and finally the fourth equation of the initial system from the obtained equation, we obtain the solution to the problem:

$$f = 293; s = 316; t = 289; r = 322.$$

Exercises 1.7

C. Karim, Dania and Tariel want to know their weight. But, unfortunately, only old scales are available, on which only objects weighing more than 100 kilograms can be weighed. Thinking, they found a way out of the situation: they weighed in pairs, and then, using the results, determined their weights. How much did each weigh if the total weight of Karim and Dania is 125 kg, Karim and Tariel – 139 kg, Dania and Tariel – 122 kg?

H. There are four bags of sugar. After weighing them in three packets, Avina received the following numbers: 3007 g, 3000 g, 2997 g, 3005 g. Determine the weight of each package, as well as the average weight of these packages.

1.8. Other method 2

Problem

Valeria decided to give notepads, albums and notebooks for 27 students of class 2A, 25 students of class 2B and 28 students of class 2C. If the students of class 2A will be given notepads, 2B – albums, 2C – notebooks, then she will spend 3626 soms. If students of class 2A will be given notebooks, 2B – notepads, 2C – albums, then she will spend 3631 soms. If students of class 2A will be presented with albums, 2B with notebooks, 2C with notepads, she will spend 3623 soms. Determine the price of a notepad, album and notebook.

Solution

Denoting the price of a notepad by n, an album by a, a notebook by b, we get a system of three equations with three unknowns: $\begin{cases} 27n + 25a + 28b = 3626; \\ 27b + 25n + 28a = 3623; \\ 27a + 25b + 28n = 3631. \end{cases}$

Pay attention to the coefficients of the unknown. They are the same. Therefore, adding all the equations of the system, we get: $27(n + b + a) + 25(n + b + a) + 28(n + b + a) =$ $= 3626 + 3631 + 3623 \Leftrightarrow 80(n + b + a) = 10880 \Leftrightarrow$ $\Leftrightarrow (n + b + a) = 136.$

Now you can get rid of one of the variables.

For example, you can subtract $27(n + b + a) = 3672$ from the first equation of the system and $25(n + b + a) = 3400$ from the second.

As a result, we get: $\begin{cases} -2a + b = -46, \\ 3a + 2b = 223. \end{cases}$

This system is easily solved, for example, by the Cramer method. Determinant of the coefficients matrix:

$$\Delta = \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} = -4 - 3 = -7.$$

Further,

$$\Delta_a = \begin{vmatrix} -46 & 1 \\ 223 & 2 \end{vmatrix} = -92 - 223 = -315.$$

Therefore, $a = -315/(-7) = 45$.

Then, from the equation $-2a + b = -46$, we get $b = 44$, and finally, from the equation $n + b + a = 136$, we get $n = 47$.

Exercises 1.8

Solve the following systems of equations:

C. $\begin{cases} 17x + 19y + 14z = 1936; \\ 17z + 19x + 14y = 1557; \\ 17y + 19z + 14x = 1507. \end{cases}$

H. $\begin{cases} 311x + 189y = 26586, \\ 189x + 311y = 23414. \end{cases}$

Summary

1–7. Solve the following systems of equations:

1) $\begin{cases} -3x + 5y = 26, \\ 8x - 7y = -44. \end{cases}$ 2) $\begin{cases} 2x - 3y + 1=0, \\ 3x + 4y - 2 = 0. \end{cases}$

3) $\begin{cases} 21x + 25y = 2020, \\ 28x + 33y = 2672. \end{cases}$ 4) $\begin{cases} 3x - 4y - 1,3 = 0, \\ 4x + 5y = 10. \end{cases}$

5) $\begin{cases} 32x + 51y = 667, \\ 18x - 23y = -90. \end{cases}$ 6) $\begin{cases} -71x + 35y = -77, \\ 28x - 18y = -55. \end{cases}$

7) Alisher decided to buy notepads for 23 girls and drawing albums for 17 boys from the orphanage. But he did not have the required $66.9. Therefore, he bought albums for girls and notepads for boys, spending $65.1. Determine the price of the album and the price of the notepad.

8) Islom plans to buy pens, rulers, and notebooks for 17 students of the class 1A, 18 students of the class 1B, and 15 students of the class 1C. If students of the class 1A will be given pens, 1B — rulers, 1C — notebooks, then he will spend $16.61. If students of grade 1A will be given notebooks,

1B — pens, 1C — rulers, then he will spend $16.58. If the students of 1A class will be given a ruler, 1B — notebooks, 1C — pens, then he will spend $16.81. Determine the price of a pen, ruler and notebook.

§2. Vectors. Vectors in a rectangular coordinate system

2.1. Definitions

A scalar is a quantity expressed by a single real number. An examples of a scalar are product price, animal weight, air temperature, etc. A scalar has magnitude, or the size of a mathematical object.

A vector is a quantity that is defined by multiple scalars. In addition to magnitude, a vector also has a direction. A vector is quantity having direction as well as magnitude, especially as determining the position of one point in space relative to another.

According to mathematicians, a vector is a directed segment. Vectors in the world around us are found everywhere. We regularly use vectors and very often do not even think about it. For example, Krylov's famous fable "Swan, Cancer and Pike" is an excellent illustration of the rule that mutually directed vectors cancel each other out. Another statement that in order to succeed, one needs desires to coincide with possibilities, in the vector language there is a statement that the sum of two vectors has the greatest length if the directions of the vectors coincide.

Vectors are widely used in physics. At the same time, the use of the Cartesian coordinate system makes vectors very useful in other areas of science.

A vector is uniquely determined by two points: the beginning and the end. The notation $\overline{AB}$ tells us that the vector $\overline{AB}$ starts at point *A* and ends at point *B*.

In order to determine the coordinates of the vector you need to subtract the coordinates of the initial point from the corresponding coordinates of the terminal point.

Problem

Determine the coordinates of the vectors $\overline{AB}$ *and* $\overline{BA}$ *defined by points A (7, –1) and B (2, 6).*

Solution

As already mentioned, to determine the coordinates of a vector, you need to subtract the coordinates of initial point *A* from the corresponding coordinates of terminal point *B*:

$\overline{AB} = (2 - 7; 6 - (-1)) = (-5; 7).$

Vector $\overline{BA} = (7 - 2; -1 - 6) = (5; -7).$

Vector $\overline{BA}$ — a vector that starts at point *B* and ends at point *A* is called the <u>opposite</u> of a vector $\overline{AB}$. It is easy to understand that the result obtained above also holds in the general case: In order to obtain the coordinates of the opposite vector, it is enough to change the signs of the coordinates of the original vector. As a result, the notation naturally turns out:

The vector opposite to the vector $\overline{a}$ is denoted by $-\overline{a}$

Exercises 2.1

C. The points *E (– 2, 1)*, *F (3, 5)*, *G (1, – 2)* are given. Determine the coordinates of the vectors $\overline{EF}$, $\overline{FE}$, $\overline{EG}$, $\overline{FG}$.

H. The points *K (– 3, – 1)*, *L (2, 4)*, *M (3, – 3)* are given. Determine the coordinates of the vectors $\overline{KL}$, $\overline{LM}$, $\overline{ML}$, $\overline{KM}$.

2.2. Sum and difference of vectors

We turn again to exercises 2.1. For clarity, we will draw the corresponding figure 2.1.

A vector $\overline{EG}$ simply means the displacement from a point *E* to the point *G*. Now consider a situation that a girl moves

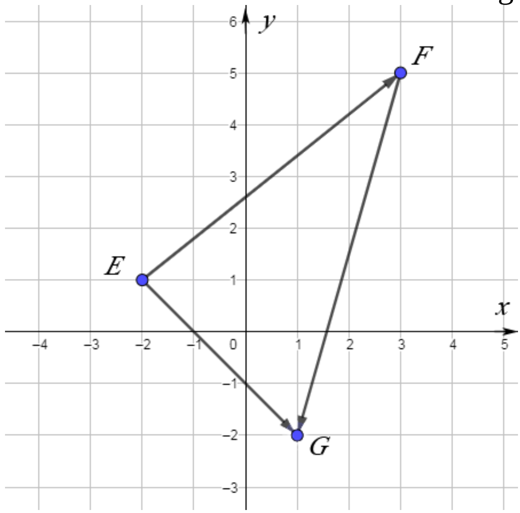


Figure 2.1

from *E* to *F* and then from *F* to *G*. The net displacement made by the girl from point *E* to the point *G*, is given by the vector $\overline{EG}$. This fact is an illustration of the vector addition rule.

If we have two vectors $\bar{a}$ and $\bar{b}$, then to add them, they are positioned so that the initial point of one coincides with the terminal point of the other. Then, the vector $\bar{a} + \bar{b}$, represented by the third side of the triangle, gives us the <u>sum (or resultant) of the vectors</u> $\bar{a}$ and $\bar{b}$.

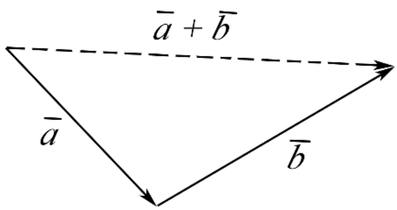


Figure 2.2

This is known as the triangle law of vector addition. In order to add several vectors, you need to do the following: find a vector that is the sum of the first two vectors and add a third to it, etc.

So $\overline{EF} + \overline{FG} = \overline{EG}$. The most surprising thing in this situation can be found by comparing the coordinates of these vectors: *(5; 4) + (– 2; – 7) = (3; – 3)*.

The coordinates of the sum of vectors are equal to the sum of the corresponding coordinates of their terms.

After we have learned to add vectors, the subtraction operation becomes very simple.

In order to subtract a vector $\bar{b}$ from a vector $\bar{a}$, you need to add the vector $\bar{a}$ to the vector $-\bar{b}$ (vector opposite to the vector $\bar{b}$).

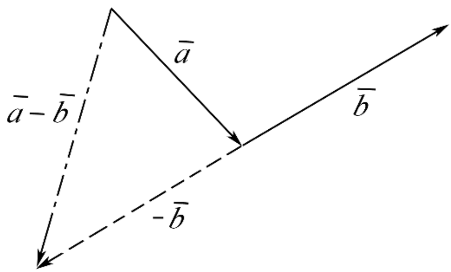


Figure 2.3

If vectors are given by coordinates, in order to add (subtract) the vectors, it is enough to add (subtract) the corresponding coordinates.

Problem

1) Vectors $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ *are given. Express the following vectors using them:*

a) $\overline{DC}$*;* *b)* $\overline{AD}$*;* *c)* $\overline{AC}$*;* *d)* $\overline{DB}$.

2) The coordinates of the points A(7, – 1), B(2, 6), C(6, 6), D(8, 4) are given. Find the coordinates of the vectors:
$\overline{AB}$, $\overline{BC}$, $\overline{CD}$, *and*
a) $\overline{DC}$; *b)* $\overline{AD}$; *c)* $\overline{AC}$; *d)* $\overline{DB}$.
3) Compare the results of problems 1 and 2.

Solution

1) The problem will be simplified if we draw a picture. (We used the coordinates of the points from problem 2, but this does not affect the solution. Any quadrilateral *ABCD* can be considered.)

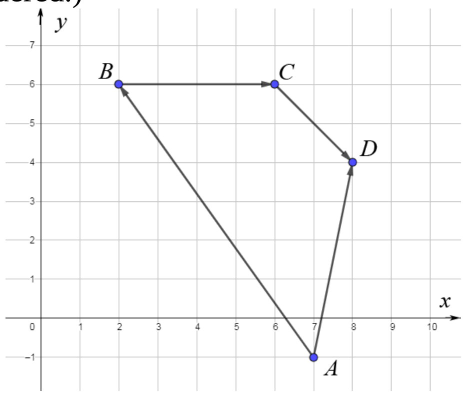


Figure 2.4

a) Vector $\overline{DC}$ is opposite to vector $\overline{CD}$. So $\overline{DC} = -\overline{CD}$.
b) From point *A* you can get to point *D*, moving sequentially on the sides *AB*, *BC*, *CD* or on the side *AD*. So
$\overline{AD} = \overline{AB} + \overline{BC} + \overline{CD}$.
c) From point *A*, you can get to point *C*, moving sequentially along the sides *AB*, *BC* or along the diagonal *AC*. So
$\overline{AC} = \overline{AB} + \overline{BC}$.
d) From point *D* you can get to point *B*, moving sequentially on the sides of *DC*, *CB*. Corresponding vectors are opposite to vectors $\overline{CD}$ and $\overline{BC}$. So
$\overline{DB} = (-\overline{CD}) + (-\overline{BC})$.

2) In order to determine the coordinates of the vector from the coordinates of the terminal point subtract the corresponding coordinates of the initial point. So

$\overline{AB} = (-5;\ 7)$, $\overline{BC} = (4;\ 0)$, $\overline{CD} = (2;\ -2)$,

a) $\overline{DC} = (-2;\ 2)$; *b)* $\overline{AD} = (1;\ 5)$;

c) $\overline{AC} = (-1;\ 7)$; *d)* $\overline{DB} = (-6;\ 2)$.

3) In order to make sure that in problems 1 and 2 the same results are obtained, we present vector equalities in coordinate form

a) $\overline{DC} = -\overline{CD} \Leftrightarrow (-2;\ 2) = -(2;\ -2)$;

b) $\overline{AD} = \overline{AB} + \overline{BC} + \overline{CD} \Leftrightarrow$

$\Leftrightarrow (1;\ 5) = (-5;\ 7) + (4;\ 0) + (2;\ -2)$;

c) $\overline{AC} = \overline{AB} + \overline{BC} \Leftrightarrow (-1;\ 7) = (-5;\ 7) + (4;\ 0)$;

d) $\overline{DB} = (-\overline{CD}) + (-\overline{BC}) \Leftrightarrow (-6;\ 2) = (-2;\ 2) + (-4;\ 0)$.

A direct calculation confirms that true equalities hold. So, for example, in case b) there are valid equalities

$1 = -5 + 4 + 2$ and $5 = 7 + 0 + (-2)$.

Exercises 2.2

C. 1) Vectors $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HI}$ are given.

Express the following vectors using them:

a) $\overline{EG}$; *b)* $\overline{FI}$; *c)* $\overline{GF}$; *d)* $\overline{EI}$.

2) The coordinates of the points

$E(-2,\ 1)$, $F(3,\ 5)$, $G(1,\ -2)$, $H(0,\ -2)$, $I(-1,\ -1)$

are given. Find the coordinates of the vectors :

$\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HI}$, and

a) $\overline{EG}$; *b)* $\overline{FI}$; *c)* $\overline{GF}$; *d)* $\overline{EI}$.

3) Compare the results of problems 1 and 2.

H. 1) Vectors $\overline{KL}$, $\overline{LM}$, $\overline{MN}$, $\overline{NO}$ are given.

Express the following vectors using them:

a) $\overline{NM}$; *b)* $\overline{LN}$; *c)* $\overline{LO}$; *d)* $\overline{KO}$.

2) The coordinates of the points $K(-4, 2)$, $L(1, 3)$, $M(3, -3)$, $N(1, -2)$, $O(-4, 5)$ are given.
Find the coordinates of the vectors: $\overline{KL}$, $\overline{LM}$, $\overline{MN}$, $\overline{NO}$, and *a)* $\overline{NM}$; *b)* $\overline{LN}$; *c)* $\overline{LO}$; *d)* $\overline{KO}$.
3) Compare the results of problems 1 and 2.

2.3. Free vector

The vectors defined above are such that any of them may be subject to its parallel displacement without changing its magnitude and direction. Such vectors are called free vectors.

So the vector defined by points $C(-5, 12)$ and $D(-3, -3)$ is the same vector as defined by points $E(105, 120)$ and $F(107, 105)$. (Make sure by determining the coordinates.) That is, vectors of the same length and direction do not differ. Vectors understood in this sense are usually denoted in small Latin letters with a dash over $\overline{a}$. In order to emphasize this feature of vectors they are called free.

It is important to note that the coordinates of the vector $\overline{a}$ can be considered as the coordinates of the point at which the end of the vector will be, if its beginning is combined with the origin.

Problem

Determine the coordinates of the point G, if known that the point H has coordinates (3, 17), and the coordinates of the vector $\overline{GH}$ *are* $(5; -12)$.

2) Determine the coordinates of the fourth vertex of the parallelogram KLMN, if known that the coordinates of the vertices points are: $K(-3, 2)$, $L(2, 4)$, $M(1, -2)$.

Solution

1) In order to find the coordinates of the point G, we assign the vector $\overline{HG}$ to the point H

Since the vector $\overline{HG}$ is the opposite of the vector $\overline{GH}$:

$\overline{HG} = -\overline{GH} = (-5; 12).$

So $G = H + \overline{HG} = (3; 17) + (-5; 12) = (-2; 29).$

2) A parallelogram is a quadrilateral in which the opposite sides are equal and parallel. Therefore, its opposite sides can be considered as vectors that are equal to each other. So

$\overline{KL} = \overline{NM}$. ($\overline{NM}$, not $\overline{MN}$.)

Because $\overline{KL} = (2-(-3); 4-2) = (5; 2) = \overline{NM}$, from the problem 1) $N = M + \overline{MN} = (1; -2) + (-5; -2) = (-4; -4).$

Exercises 2.3

C. 1) Determine the coordinates of the point T, if known that the point S has coordinates $(-5.2, 7)$, and the coordinates of the vector $\overline{ST}$ are $(6; -1/7)$.

2) Determine the coordinates of the fourth vertex of the $PQRT$ parallelogram, if known that the coordinates of the remaining vertices are: $P(-5, -2), Q(-2, 3), R(1, -2).$

H. 1) Determine the coordinates of the point P, if known that the point G has the coordinates (–9, 3), and the coordinates of the vector $\overline{GP}$ are $(2; -11)$.

2) Determine the coordinates of the fourth vertex of the parallelogram $ABCD$, if known that the coordinates of the remaining vertices are: $A(-6, -3), B(-4, 2), C(3, -5).$

2.4. Length of the vector

Problem

Let $\overline{AB}$ be a vector where A (2, – 1) and B (14, 4). Find its length.

Solution

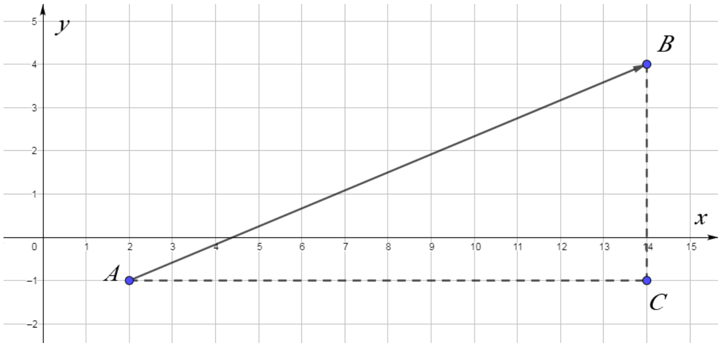


Figure 2.5

The figure shows that the length of the vector is the hypotenuse of the triangle *ABC*. It is easy to understand that, since the catheti of the *AC* and *BC* are parallel to the coordinate axes, their lengths are equal to the difference of the corresponding coordinates. So

$|AC| = |14 - 2| = 12; \quad |BC| = |4 - (-1)| = 5.$

Then, by the Pythagorean theorem,

$$|AB| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13.$$

It should be noted that the numbers 12 and 5 are the coordinates of the vector, and our calculations confirm the rule:

The length of the vector is the square root of the sum of the squares of its coordinates.

Exercises 2.4

C. Find the length of the vector $\overline{EF}$, if $E(-2, 1), F(3,-2)$.

H. Find the length of the vector $\overline{AR}$, if $A(-4, 2), R(6, 3)$.

2.5. Multiplication of a vector by a scalar

In cases where they change the length of the vector and do not change its direction, they talk about multiplying the vector by a positive number. If the length of the vector changes, and the direction of the vector changes to the opposite, they say that the vector is multiplied by a negative number. In order to multiply a vector by a number *k*, you need to increase its length *k* times, and: keep the direction if *k* is positive; reverse direction if *k* is negative. When using coordinate notation, in order to multiply the vector by the number *k* (change its length by *k* times), it is enough to multiply each coordinate by *k*.

Problem

Let $\overline{CD}$ *be a vector where C (– 1, 5) and D (3, 2). It is required to multiply it by (– 3) and calculate the length of the resulting vector.*

Solution

The vector $\overline{CD} = (3 – (– 1); 2 – 5) = (4; \ – 3)$. Then the vector $– 3\overline{CD} = (– 12; 9)$.

We need to calculate the length. There are two approaches to do this.

You can calculate the length of the vector $\overline{CD}$:

$\left|\overline{CD}\right| = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = 5$, and triple the result: $5 \cdot 3 = 15$.

Another way: calculate the length of the resulting vector $-3\overline{CD}$: $\left|-3\overline{CD}\right| = \sqrt{(-12)^2 + 9^2} = \sqrt{144+81} = \sqrt{225} = 15$.

Exercises 2.5

C. Find the length of the vector $4\overline{UW}$, if $U(–2, 6), W(4, 0)$.

H. Find the length of the vector $2\overline{TA}$, if $T(–1, 3), \ A(2, 1)$.

2.6. Unit vector

Vectors whose length is equal to one are called unit or normalized.

Problem

Let $\overline{YZ}$ be a vector, where Y (– 3, 5) and Z (5, – 3). Determine the coordinates of the unit vector.

Solution

Vector $\overline{YZ}$ = $(5-(-3); (-3)-5) = (8; -8)$.

Length of vector

$$\overline{YZ} = \sqrt{8^2+(-8)^2} = \sqrt{64+64} = \sqrt{128} = 8\sqrt{2}.$$

In order for a unit vector it is enough to divide the coordinates of the original vector by its length.

So, the coordinates of the unit vector are: $\widehat{YZ} = (8/8\sqrt{2};\ -8/8\sqrt{2}) = (1/\sqrt{2};\ -1/\sqrt{2})$.

Exercises 2.6

C. Let $\overline{JH}$ be a vector, where *J (–1, 15)* and *H (6, –9)*. Determine the coordinates of the unit vector.

H. Let $\overline{AC}$ be a vector, where *A(–4, 2)* and *C(2, 10)*. Determine the coordinates of the unit vector.

2.7. Parallel vectors

If the vectors are parallel, then their directions are the same or opposite. Therefore, in order to get another one parallel to it from one vector, it is enough to multiply it by the corresponding number.

Problem

Determine the coordinates of a vector $\overline{a}$ that is parallel to the vector (5; –12) and has a length of 5.

Solution

We calculate the length: $\overline{a} = \sqrt{5^2+(-12)^2} = \sqrt{169} = 13$. Lets divide its coordinates by 13, we get a unit vector (5/13; –12/13), then multiplying the result by 5, we get a vector with a length of 5. This vector $\left(\frac{25}{13}; \frac{-60}{13}\right)$, as well as the vector opposite to it $\left(-\frac{25}{13}; \frac{60}{13}\right)$, will be solutions to the problem.

Exercises 2.7

C. Determine the coordinates of a vector $\bar{b}$ that is parallel to the vector *(–9; 40)* and has a length of 10.

H. Determine the coordinates of a vector $\bar{c}$ that is parallel to the vector *(–5; 12)* and has a length of 4.

2.8. Dividing a segment in a given ratio

The conclusions from the previous examples allow us to obtain a solution to the problem of dividing a segment in a given ratio. So, lets determine the coordinates of the point $C(x_c; y_c)$, which divides the segment AB in the ratio ***s: k***. For the solution, it is enough to notice that the vector $\overline{AC}$ is parallel to the vector $\overline{AB}$ and has the length $\frac{s}{s+k}\,|\overline{AB}|$.

Problem

Find the coordinates of point C, which divides the segment AB in a ratio 3: 2, where point A has coordinates (3, 17), point B has coordinates (9, 9).

Solution

To find the coordinates of point *C*, we define the coordinates of the vector $\overline{AB}$: *(9 – 3; 9 – 17) = (6; – 8)*, then the coefficient $\frac{s}{s+k} = \frac{3}{3+2} = 0.6$ and the coordinates of the vector $\overline{AC}$: *0.6(6; – 8) = (3.6; – 4.8)*. Next, we use the fact that the coordinates of point *A* and vector $\overline{OA}$, as well as point C and vector $\overline{OC}$ coincide. Therefore, since $\overline{OA} + \overline{AC} = \overline{OC}$, we find the coordinates of the vector $\overline{OC}$ — coordinates of the point *C*: $x_C = 3 + 3.6 = 6.6$; $y_C = 17 + (-4.8) = 12.2$.

The most common problem is to divide the vector in half. In this case, $\frac{s}{s+k}=\frac{1}{1+1}$ $= 0.5$.

Then, the midpoint of the vector $\overline{AB}$ will have the coordinates

$$(x_A; y_A)+0.5(x_B-x_A;\ y_B-y_A)=$$
$$=\left(x_A+0.5(x_B-x_A);\ y_A+0.5(y_B-y_A)\right)=$$
$$=\left(0.5(x_A+x_B);\ 0.5(y_A+y_B)\right).$$

We have shown that the coordinates of the midpoint of the vector (segment) are equal to the half-sum of the coordinates of its ends.

In particular, in the conditions of the problem, the coordinates of the midpoint of the vector $\overline{AB}$ are *(6; 13).*

Exercises 2.8

C. Find the coordinates of the point *R*, which divides the segment *PQ* in the ratio 1: 3, where the point *P* has coordinates *(– 3, 7)*, the point *Q* has coordinates *(5, 1).*

H. Find the coordinates of point *H*, which divides the segment *CF* in the ratio 2: 1, where point *C* has coordinates *(6, –7)*, point F has coordinates *(3, 2).*

2.9. Problem of the sum of vectors

Problem

Zhanik and Aybike kicked the ball at the same time. If only Zhanik had done this, the ball would have flown north at a speed of 14 m / s. If only Aybike had done this, the ball would have flown northwest at a speed of $10\sqrt{2}$ *m / s. At what distance will the ball be in 3 seconds? (To simplify the situation, neglect air resistance.)*

Solution

If we take the place of impact on the ball as the origin and assume that the north direction is determined by the axis of the *OY*, then the kick on the ball of Zhanik is expressed by the vector $\overline{a}\,(0;\ 14)$. The corresponding vector $\overline{b}$ for Aybike will turn out if we consider a square with a diagonal of $10\sqrt{2}$. Let's draw a picture illustrating Aibike's kick.

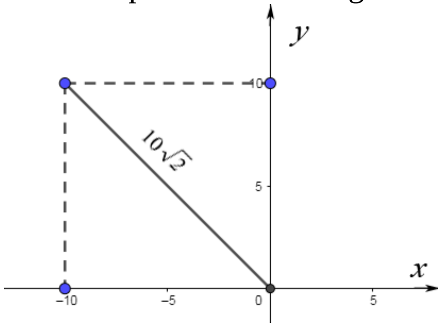


Figure 2.6

Now, by the Pythagorean theorem, we obtain that the coordinates of the vector $\overline{b}$ are *(– 10; 10)*. The result of a simultaneous hit on the ball is the sum of the vectors: $\overline{a}+\overline{b}$ =*(– 10; 24)*. Its length $\sqrt{10^2+24^2}=\sqrt{100+576}=26$ expresses the distance that the ball flew in 1 second.

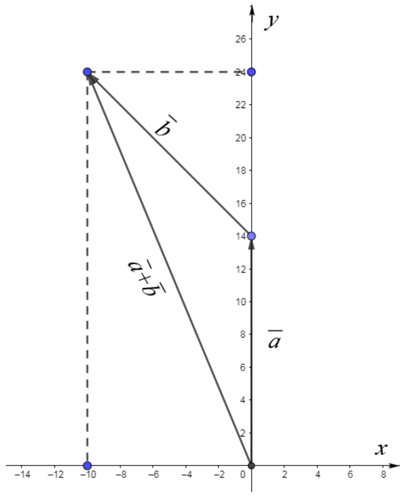


Figure 2.7

Therefore, if there were no air resistance, in 3 seconds the ball would fly 78 meters.

Exercises 2.9

C. Kanai and Rasul both kicked the ball. If only Kanai had done this, the ball would have flown south at a speed of 15 m / s. If only Rasul did this, then the ball would fly west at a speed of 8 m / s. At what distance will the ball be in 4 seconds (To simplify the situation, neglect the air resistance.)?

H. Aybek and Malika kicked the ball at the same time. If only Aybek had done this, the ball would have flown north at a speed of 5 m / s. If only Malika had done this, the ball would have flown east at a speed of 12 m / s. At what distance will the ball be in 5 seconds from them (To simplify the situation, neglect the air resistance.)?

2.10. Area of triangle

Using Cartesian coordinates can also help in solving classical geometry problems, which include the problem of calculating the area of a triangle. In those cases when the lengths of all sides are given, the Heron formula is used to solve such a problem. We propose a different approach

Problem

Find the area of a triangle with sides 17 cm, 25 cm, 28 cm.

Solution

Consider the Cartesian coordinate system on the plane. We position the longest side of the triangle on the *OX* axis so that one of the vertices appears at the point *(0, 0)* at the origin, and the other at *(28, 0)*. The coordinates of the third vertex are denoted by (x_A, y_A).

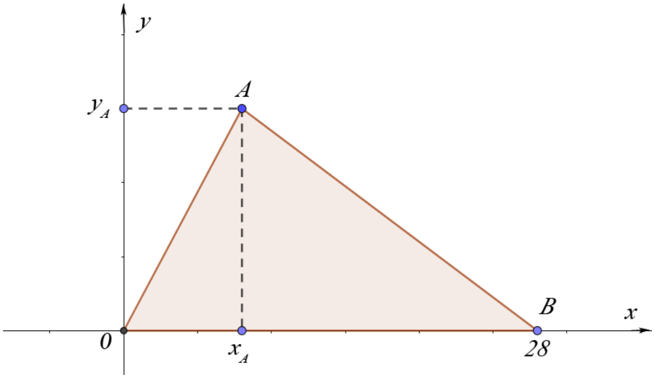


Figure 2.8

We use the formula for finding the length of the segment (vector) and get the system:

$$\begin{cases} (x_A - 0)^2 + (y_A - 0)^2 = 17^2; \\ (x_A - 28)^2 + (y_A - 0)^2 = 25^2; \end{cases} \Leftrightarrow \begin{cases} x_A{}^2 + y_A{}^2 = 17^2; \\ (x_A - 28)^2 + y_A{}^2 = 25^2. \end{cases}$$

Now subtract the 2nd equation from the 1st equation of the system: $x_A{}^2 - (x_A - 28)^2 = 17^2 - 25^2$.

To simplify this equation, it is appropriate to use the square difference formula $a^2 - b^2 = (a - b)(a + b)$.Then:

$x_A{}^2 - (x_A - 28)^2 = 17^2 - 25^2 \Leftrightarrow 28(2x - 28) = -8 \cdot 42 \Leftrightarrow$
$\Leftrightarrow (2x - 28) = -8 \cdot 42/28 \Leftrightarrow 2x_A = -12 + 28 \Leftrightarrow x_A = 16/2 = 8.$

Substitute the found value in the 1st equation of the system:
$8^2 + y_A{}^2 = 17^2 \Leftrightarrow y_A{}^2 = 225.$

As a result, we got two solutions – two points at which the 3rd vertex of the triangle can be located:
$(8, 15)$ or $(8, -15)$.

The figure shows that our triangle can be considered as the union of two rectangular triangles with cathetus 8 and $28 - 8 = 20$, and common cathetus 15.

Therefore, the sum of their areas: $8 \cdot 15/2 = 60;$ $20 \cdot 15/2 = 150$ will be the desired number:
$60 + 150 = 210.$

Exercises 2.10

C. Calculate the area of a triangle with sides *20 mm*, *65 mm*, *75 mm.*

H. Calculate the area of a triangle with sides *101 cm*, *25 cm*, *114 cm.*

2.11. Scalar (dot) product of vectors

A scalar product of vectors – is a number, which equals to the product of lengths of these vectors to the cosine angle between them. The scalar product is denoted as $(\bar{a}\,|\,\bar{b})$ or $(\bar{a}, \bar{b})$ or $\langle \bar{a}, \bar{b} \rangle$ or $\bar{a} \cdot \bar{b}$. We think that the first notation is more preferable, because others are often used in denoting other objects. So,

$$(\bar{a}\,|\,\bar{b}) = |\bar{a}| \cdot |\bar{b}| \cos \alpha .$$

According to the definition, the scalar product of the perpendicular vectors equals zero.

Just like in previous calculations, application of the coordinate allows us to simplify the process of calculation of scalar product: in order to do that it is suffice to calculate the sum of the product of the respective coordinates.

$$(\bar{a}\,|\,\bar{b}) = x_a \cdot x_b + y_a \cdot y_b.$$

Problem

A shop sells 4 types of products.

Let the vector $\bar{b}\,(102; 205; 283; 792)$ *express sales volume.*

Solution

If the prices of these products are determined by the vector $\bar{p}\,(200; 20; 35;120)$, then the revenue is the scalar product of the vectors :

$\bar{p}\,(200;\ 20;\ 35;\ 120)$ and $\bar{b}\,(102; 205; 283; 792)$:

$R = (\bar{p}\,|\,\bar{b}) = 200 \cdot 102 + 20 \cdot 205 + 35 \cdot 283 + 120 \cdot 792 =$
$= 129445.$

Exercises 2.11

C. Points $A(-2,\ 1,\ 3)$, $B(1, -1,\ 5)$ and vector $\bar{a}(-5;\ 2;\ 4)$ are given. Find scalar product of the vectors $2\overline{AB}$ and $\bar{a}$.

H. Points $A(2,\ -3)$, $B(1, 5)$ and vector $\bar{a}(-5;\ 1)$ are given. Find scalar product of the vectors $\overline{BA}$ and $0.5\bar{a}$.

2.12. Angle between the two vectors

One of the popular problems is the problem to define the angle between the vectors. As we already know, the value of the scalar product can be found as the product of the lengths of these vectors to the cosine of the angle between them, and also as the sum of the product of the respective coordinates, so we can write and solve the respective equation.

Problem

Let's assume that the following vectors are given $\bar{a}(3;\ 0)$ and $\bar{b}(6;\ 2\sqrt{3})$. Then $(\bar{a}\,|\,\bar{b}) = 3\cdot 6 + 0\cdot 2\sqrt{3} = 18;$

$|\bar{a}| = \sqrt{3^2+0^2} = \sqrt{9} = 3;$

$|\bar{b}| = \sqrt{6^2+(2\sqrt{2})^2} = \sqrt{48} = 4\sqrt{3}.$

Hence, $18 = 3\cdot 4\sqrt{3}\ \cos\alpha.$

Therefore $\cos\alpha = \dfrac{18}{3\cdot 4\sqrt{3}} = \dfrac{\sqrt{3}}{2}$, and $\alpha = 30^0$.

Exercises 2.12

C. Points $A(-2,\ 1,\ 3)$, $B(1, -1, 5)$ and vector $\bar{a}\ (-5;\ 2;\ 4)$ are given. Find the angle between the vectors $\overline{BA}$ and $\bar{a}$.

H. Points $A(2,\ -3)$, $B(1, 5)$ and vector $\bar{a}(-5;\ 1)$. Find the angle between the vectors $\overline{AB}$ and $2019\bar{a}$.

2.13. Vector decomposition

Many problems which involve setting up equations can be interpreted as problems on vector decomposition – finding the coordinates of n-dimensional vector $\overline{a}$ in a basis of vectors $\overline{b^1}$, $\overline{b^2}$, $\overline{b^3}$, ..., $\overline{b^n}$. As the matter of fact, this is a problem on solving the system of n linear equations with n unknowns:

$$\overline{a} = x_1 \overline{b^1} + x_2 \overline{b^2} + x_3 \overline{b^3} + \ldots + x_n \overline{b^n}.$$

Problem

Find the coordinates of the vector (159; 172) on the basics vectors (0.3; 0.4) and (0.7; 0.6).

Vector equation $(159;\ 172) = x(0.3;\ 0.4) + y(0.7;\ 0.6)$ can be rewritten in the form of the system: $\begin{cases} 0.3x + 0.7y = 159, \\ 0.4x + 0.6y = 172. \end{cases}$

System's solution: $x = 250;\ y = 120$ are the required coordinates.

This problem can be considered as a mathematical model of the following problem:

A shop sells two types of candy mixes. The first one contains 300 grams of chocolate candy, 700 grams of caramel candy and costs 159 soms. The second one, contains 400 grams of chocolate candy, 600 grams of caramel candy and costs 172 soms. How much does 1 kilogram of chocolate candy cost and how much does 1 kilogram of caramel candy cost?

Hint

The problem on vector decomposition has solution only in situation when a determinant, the columns of which are the coordinates of the basic vectors $\overline{b^1}$, $\overline{b^2}$, $\overline{b^3}$, ..., $\overline{b^n}$, is nonzero.

Exercises 2.13

C. Points $A(1,\ 3)$, $B(-1,\ 5)$, $C(3, -2)$ and vector $\overline{a}(-5;4)$

are given. Find the coordinates of the vector $\overline{a}$ on the basis of the vectors $\overline{AB}$, $\overline{AC}$.

H. Points $A(2, -3), B(1, 5)$, $C(-2,\ 7)$ and vector $\overline{a}\,(5; -3)$ are given. Find the coordinates of the vector $\overline{a}$ on the basis of the vectors $\overline{BA}$, $\overline{BC}$.

2.14. Cartesian components of vectors.

Unit vectors may be used to represent the axes of a Cartesian coordinate system. So,

i is a unit vector in the direction of the OX – axis;
j is a unit vector in the direction of the OY – axis;
k is a unit vector in the direction of the OZ – axis.

They are also often denoted using common vector notation $(\overline{i}, \overline{j}, \overline{k})$ rather than standard unit vector notation $(\hat{\imath}, \hat{\jmath}, \hat{k})$. In most contexts it can be assumed that ***i, j,*** and ***k,*** are versors of a 3D Cartesian coordinate system.

Therefore, notations $\overline{a}\ (2; -3; 4)$ and $\overline{a} = 2i - 3j + 4k$ are equivalent.

Problem

Write the vector $3\overline{a} - 2.5\overline{b}$ *, in terms of the unit vectors **i** and **j**, if* $\overline{a} = -2.2i + 3j;\ \overline{b} = 8.5i + 2.3j.$

Find scalar product of the vectors $\overline{a}$ *and* $\overline{b}$ *.*

Solution

Since we still used a different form of writing vectors, it will be logical to go to it: $3\overline{a} - 2.5\overline{b} =$
$=3(-2.2; 3) - 2.5(8.5;\ 2.3) =(-6.6;\ 9) - (21.25;\ 5.75) =$
$= (-27.85;\ 3.25) = -27.85i + 3.25j.$

Scalar product: $(\overline{a}\,|\,\overline{b}\,) = (-2.2i + 3j)|(8.5i + 2.3j)=$
$= -18.7(i|i)+25.5(j|i) -5.06(i|j)+6.9(j|j) =$
$=[(i|i)=1,\ (i|j) = (j|i) = 0,\ (j|j)=1] = -18.7 + 6.9 = -11.8.$

Exercises 2.14

C. Vectors $\bar{a} = 12i - 3j - 5k$; $\bar{b} = -5i + 2j + 7k$ are given. Write the vector $1.3\bar{a} + 2\bar{b}$, in terms of the unit vectors ***i*** and ***j***. Find scalar product of the vectors $\bar{a}$ and $\bar{b}$.

H. Vectors $\bar{a} = -9i + 4j$; $\bar{b} = 3i + 1.2j$ are given. Write the vector $-0.4\bar{a} + 6\bar{b}$, in terms of the unit vectors ***i*** and ***j***. Find scalar product of the vectors $\bar{a}$ and $\bar{b}$.

2.15. Problem.

The points A(12, 4), B(2, – 6), C(16, – 4) are given. The point P that is divide the segment AB in the ratio 1:4, the point Q that is midpoint of the segment BC and the point R that is divide the segment AC in the ratio 1: 3. Find:

a) Coordinates of the points P,Q and R.

b) Scalar product of the vectors $\overline{PQ}$ *and* $\overline{PR}$ *;*

c) Angle between vectors $\overline{PQ}$ *and* $\overline{PR}$ *;*

d) Area of a triangle PQR.

e) Coordinates of the point T that is the vertex of the parallelogram PQRT;

f) Coordinates of the vector $\overline{QT}$ *in basis of vectors* $\overline{AB}$ *and* $\overline{BC}$ *.*

Solution Let's use an illustration.

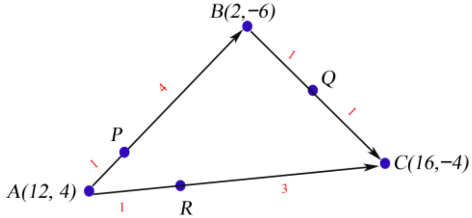


Figure 2.9

a) In order to find the coordinates of point P, we find the coordinates of the vector $\overline{AB}$:

$\overline{AB} = (2 - 12; -6 - 4) = (-10; -10).$

From figure 2.9 it is clear that

$$\overline{AP} = \frac{1}{5}\overline{AB} = \frac{1}{5}(-10; -10) = (-2; -2).$$

Suppose P has coordinates (x_P, y_P), then

$\overline{AP} = (x_P - 12; y_P - 4) = (-2; -2)$, we obtain, that

$x_P - 12 = -2 \Rightarrow x_P = 10;$

$y_P - 4 = -2 \Rightarrow y_P = 2 \quad \Rightarrow \quad P(10, 2).$

In the same way, we can find coordinates of the points Q and R.

$\overline{BC} = (16 - 2; -4 + 6) = (14; 2);$

$$\overline{BQ} = \frac{1}{2}\overline{BC} = \frac{1}{2}(14; 2) = (7; 1);$$

Suppose Q has coordinates (x_Q, y_Q), then

$\overline{BQ} = (x_Q - 2; y_Q + 6) = (7; 1)$, we obtain, that

$x_Q - 2 = 7 \Rightarrow x_Q = 9;$

$y_Q + 6 = 1 \Rightarrow y_Q = -5 \quad => \quad Q(9, -5);$

$\overline{AC} = (16 - 12; -4 - 4) = (4; -8);$

$$\overline{AR} = \frac{1}{4}\overline{AC} = \frac{1}{4}(4; -8) = (1; -2).$$

Suppose R has coordinates (x_R, y_R), then

$\overline{AR} = (x_R - 12; y_R - 4) = (1; -2)$, we obtain, that

$x_R - 12 = 1 \Rightarrow x_R = 13;$

$y_R - 4 = -2 \Rightarrow y_R = 2 \quad \Rightarrow \quad R(13, 2).$

b) We find the coordinates of the vectors

$\overline{PQ} = (-1; -7)$, $\overline{PR} = (3; 0)$, then

$$\left(\overline{PQ} \middle| \overline{PR}\right) = (-1) \cdot 3 + (-7) \cdot 0 = -3$$

c) $\left|\overline{PQ}\right| = \sqrt{(-1)^2 + (-7)^2} = \sqrt{50}, \quad \left|\overline{PR}\right| = \sqrt{(3)^2 + (0)^2} = 3.$

We know that

$$\cos P = \frac{\left(\overline{PQ}\middle|\overline{PR}\right)}{\left|\overline{PQ}\right| \cdot \left|\overline{PR}\right|} = \frac{-3}{\sqrt{50} \cdot 3} = -\frac{1}{\sqrt{50}} \Rightarrow \angle P = \arccos\left(-\frac{1}{\sqrt{50}}\right).$$

d) Let use the formula for area of triangle

$$S_{\Delta PQR} = \frac{1}{2}\left|\overline{PQ}\right| \cdot \left|\overline{PR}\right| \cdot \sin P;$$

$$\sin P = \sqrt{1 - \cos^2 P} = \sqrt{1 - \frac{1}{50}} = \sqrt{\frac{49}{50}} = \frac{7}{\sqrt{50}};$$

$$S_{\Delta PQR} = \frac{1}{2}\left|\overline{PQ}\right| \cdot \left|\overline{PR}\right| \cdot \sin P = \frac{1}{2} \cdot \sqrt{50} \cdot 3 \cdot \frac{7}{\sqrt{50}} = 10.5.$$

e) Let's use an illustration.

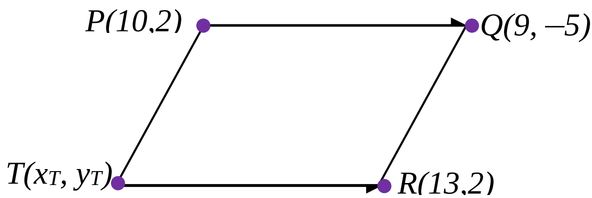


Figure 2.10

Pay attention to the order of the vertices of the parallelogram *PQRT*. Suppose *T* has coordinates (x_T, y_T), then

$\overline{PQ} = (-1; -7); \ \overline{TR} = (13 - x_T; 2 - y_T).$

Since $\left|\overline{PQ}\right| = \left|\overline{TR}\right|$ and it is known that $\overline{PQ}$ and $\overline{TR}$ are parallel and have the same direction, then $\overline{PQ} = \overline{TR}$ or

$(-1; -7) = (13 - x_T; 2 - y_T) \Rightarrow -1 = 13 - x_T \Rightarrow x_T = 14;$

$-7 = 2 - y_T \Rightarrow y_T = 9; \ \Rightarrow T(14, 9)$

f) The vector *QT* in basis of vectors $\overline{AB}$ and $\overline{BC}$:

$\overline{QT} = m\overline{AB} + n\overline{BC};$

$\overline{QT}$ =(5; 14); $\overline{AB}$ =(– 10; – 10); $\overline{BC}$ = (14; 2).

(5; 14) = m(– 10; – 10) + n(14; 2)

(5; 14) = (–10m+14n; –10m+2n) $\Rightarrow$

$$\Rightarrow \begin{cases} -10m+14n=5, \\ -10m+2n=14, \end{cases} \Rightarrow \begin{cases} m=-1.55 \\ n=-0.75 \end{cases}$$

$$\overline{QT}=-1.55\overline{AB}-0.75\overline{BC}.$$

Summary

1. Let a convex pentagon be given *ABCDE*.

Find:

a) Vector $\overline{AB}+\overline{BC}$;

b) Vector $\overline{BC}+\overline{CD}+\overline{DE}+\overline{EA}$;

c) Vector $\overline{AB}-\overline{CB}+\overline{CE}$;

d) Vector $\overline{AD}-\overline{CD}-\overline{BC}$.

2. *ABC* is a right triangle defined with *AC* and *BC* catheti, the lengths of which are 3 and 4, respectively.

Find:

a) Length of vector $2\,\overline{AC}+3\overline{CB}$;

b) Length of vector $4\,\overline{AC}-\overline{BC}$.

3 Determine *x* and *y*, if it's known that:

a) vectors *(x; 7)* and *(2; 6)* are parallel;

b) vectors *(5; x; 3)* and *(2; 6; y)* are parallel;

4. Points *A (2, –1), B (4, 1), C (2, 5)* are given. Find:

a) Coordinates and lengths of vectors $\overline{AB}$, $\overline{AC}$;

b) Coordinates of the point *D*, if it's known that the vector $\overline{BD}$ is equal to the vector $\overline{AB}$;

c) Coordinates of point *E*, if it's known that the vector $\overline{BE}$ is parallel and 2 times longer than the vector $\overline{AB}$ *;*

d) Coordinates of point *E* dividing the segment *AB* in the ratio 1: 2;

e) Scalar product of the vectors $0.5\overline{AB}$ and $4\overline{AC}$.

5. Points *A (–2, 3), B (1, –1), D(4, 1)* are given. Find:

a) Coordinates of point *C*, if the vector $2\overline{AB} - 3\overline{AC}$ has coordinates *(0; –11)*;

b) Coordinates of point E dividing the segment *AB* in the ratio 3: 2;

c) Coordinates of a vector that is parallel to the vector $\overline{AB}$ and 3 times longer;

d) Angle between the vectors $\overline{AB}$ and $\overline{AD}$.

6. Calculate the area of triangle with sides *101 cm, 25 cm, 114 cm.*

7. The points are given: *A (2, – 3), B (4, 0), C(– 5, 1).* Find:

a) Coordinates of a vector $\overline{a}$, perpendicular to the vector $\overline{AB}$, if the length of the vector $\overline{a}$ is equal to 10;

b) Coordinates of a point *K*, dividing the segment *AB* in the ratio 5: 2;

c) Angle between the vectors $\overline{AB}$ and $\overline{AC}$;

d) Length of the segment *BC*;

e)Area of a triangle *ABC*;

f) Coordinates of a vector *(10;– 12)* in basis of vectors $\overline{AB}$ and $\overline{BC}$.

8. Vectors $\overline{a} = -6i + j$; $\overline{b} = i - 7j$ are given. Write the vector $-4\overline{a} + 1.6\overline{b}$, in terms of the unit vectors ***i*** and ***j***. Find scalar product of the vectors $\overline{a}$ and $\overline{b}$.

§3. Equation of a straight line

"Mark two points on a piece of paper (on a blackboard) and connect them". We have offered this task to our students many times. Almost everyone, with rare exceptions, draws a line segment. So, a straight line is natural. Using the straight line equation allows you to understand many problems of the economics, business, etc. The fact is that, practically, any curve with a high degree of accuracy can be represented as a broken line, consisting of straight line sections.

3.1. Problem

The bus ALPHA left Bishkek for Naryn and moved at a speed of 60 km/h. At what distance from Bishkek was the bus in a) 2 hours; b) 2.5 hours; c) 3 hours 30 minutes; d) 4 hours 12 minutes? Draw a Cartesian coordinate system on the plane and mark the corresponding points on it. Make sure that you can draw a straight line through these points.

Solution

The distance will be determined by the formula $s = 60t$, where t is the time measured in hours. Substituting the time values into the formula, we will receive answers in (km):

a) $s = 60 \cdot 2 = 120;$

b) $s = 60 \cdot 2.5 = 150;$

c) Since 60 minutes is an hour, 3 hours and 30 minutes is 3.5 hours. Therefore, $s = 60 \cdot 3.5 = 210;$

d) 4 hours 12 minutes = 4.2 hours, because 12 minutes is $12/60 = 0.2$ hours. Therefore, $s = 60 \cdot 4.2 = 252.$

Thus, on the coordinate plane you need to mark the points: $(2, 120)$, $(2.5, 150)$, $(3.5, 210)$, $(4.2, 252)$ where the first coordinates indicate time and are measured on the horizontal axis $0t$. The second coordinates indicate the distance from

Bishkek in the corresponding time and are measured on the vertical axis *0s*.

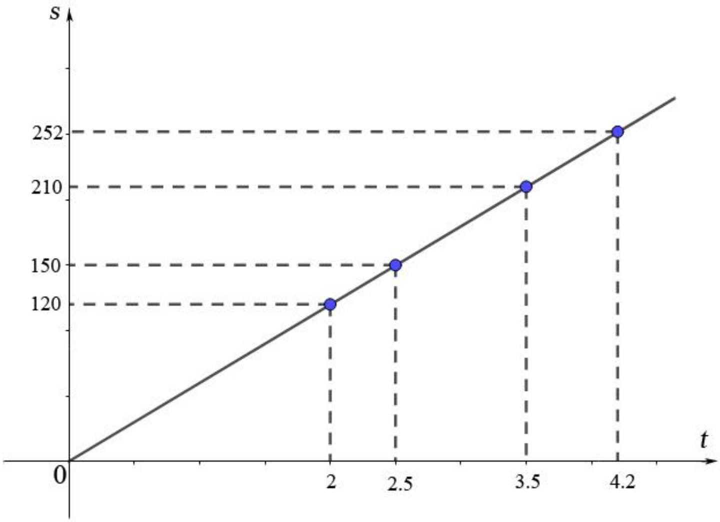


Figure 3.1

With a ruler, make sure that all these points lie on the same line.

Exercises 3.1

C. Yunus sells beans at a price of 8 lire per kg. How much money will he have after the sale of
a) 2 kg; b) 5 kg of beans? How many kg of beans Yunus sold if he has c) 24 lire; d) 30 lire? Draw a Cartesian coordinate system on the plane and mark the corresponding points on it, mark the weight of the sold beans on the horizontal axis, and the number of lire on the vertical axis. Make sure you can draw a straight line through these points. Write a function of the number of lire which depends on the weight of the beans sold.

H. Water flows into the pool at a rate of 5 liters per minute. How many liters of water will be in the pool after a) 3 minutes; b) 7 minutes? After how many minutes will be c) 20 liters; d) 14 liters of water in the pool? Draw a Cartesian coordinate system on the plane and mark the corresponding points on it, mark the number of minutes on the horizontal axis and the number of liters of water on the vertical axis. Make sure you can draw a straight line through these points. Write a function of the volume of water in the pool which depends on time in minutes.

3.2. Problem

45 minutes before the ALPHA bus (see problem 1), BETA bus left Bishkek for Naryn. Also, from Bishkek to Naryn 1.5 hours after the ALPHA bus, the GAMMA bus left. Both of these buses, as well as the ALPHA bus, moved at a speed of 60 km/h. How far from Bishkek are BETA and GAMMA after a) 2 hours; b) 2.5 hours; c) 3 hours 30 minutes; d) 4 hours 12 minutes after the departure of the ALPHA bus from Bishkek? Draw a Cartesian coordinate system on the plane and mark the corresponding points on it. Make sure that the points related to BETA can be drawn straight. The same is true for GAMMA. Draw three straight lines related to these three buses in the same coordinate plane. Make sure they are parallel.

Solution

In 45 minutes, that is, in 0.75 hours, the BETA bus drove $60 \cdot 0.75 = 45$ kilometers. Then, since the distance traveled by the ALPHA is determined by the formula $s = 60t$, the distance traveled by BETA is determined by the formula $s_B = 60t + 45$.

From the same considerations, since $60 \cdot 1.5 = 90$, the distance traveled by GAMMA is determined by the formula $s_G = 60t - 90$.

Substituting the values of time into the specified formulas, we will receive answers in (km):

a) $s_B = 60 \cdot 2 + 45 = 165$; $s_G = 60 \cdot 2 - 90 = 30$;

b) $s_B = 60 \cdot 2.5 + 45 = 195$; $s_G = 60 \cdot 2.5 - 90 = 60$;

c) $s_B = 60 \cdot 3.5 + 45 = 255$; $s_G = 60 \cdot 3.5 - 90 = 120$;

d) $s_B = 60 \cdot 4.2 + 45 = 297$; $s_G = 60 \cdot 4.2 - 90 = 162$.

Thus, the straight line describing the movement of the BETA bus passes through the points:
(2, 165), (2.5, 195), (3.5, 255), (4.2, 297), and the straight line describing the movement of the GAMMA bus passes through the points: *(2, 30), (2.5, 60), (3.5, 120), (4.2, 162).*

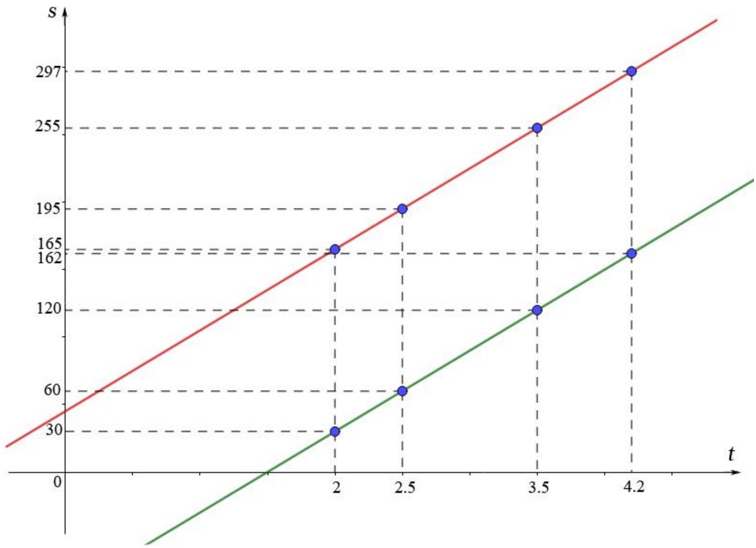


Figure 3.2

Functions: $s = 60t$, $s_B = 60t + 45$, $s_G = 60t - 90$ are special cases of a linear function, which is usually written in the form

$$y = mx + b.$$

The number b is the coordinate of the point of intersection of the straight line with the vertical axis.

The coefficient m is called the slope (angular coefficient) of a straight line.

Imagine that the ALPHA, BETA and GAMMA buses, arriving in Naryn, will not stop and continue driving at the same speeds. When will ALFA catch up with BETA and when will BETA catch up with GAMMA? The answer, of course, is obvious: never. In the language of graphs – all three straight lines have the same slope, equal to 60 — all three straight lines are parallel to each other. Since such straight lines do not intersect, graphs expressing the movement of buses cannot have common points.

Exercises 3.2

C. Answer the questions from the exercise 3.1**C**, given that at the initial time 1) Yunus had 10 lire; 2)Yunus owed 5 lire.

H. Answer the questions for exercise 3.1**H**, given that at the initial time 1) there were 2 liters of water in the pool; 2) there were 8 liters of water in the pool.

3.3. Problem

At 1:00 PM the motorcycle was at a distance of 25 km from Bishkek, at 4:00 PM — at a distance of 226 km. Determine the slope of the straight line describing the movement of the motorcycle. Draw its graph.

At 2:00 PM that same day, the cyclist was at a distance of 50 km from Bishkek, at 6:00 PM — at a distance of 138 km. Determine the slope of the straight line describing the movement of the cyclist. Draw its graph in the same coordinate plane as the motion graph of the motorcycle.

Solution

Putting the time values on the horizontal and the distances on the vertical axis, we get two points related to the motorcycle. Connecting them, we get the corresponding line. Also do with the cyclist.

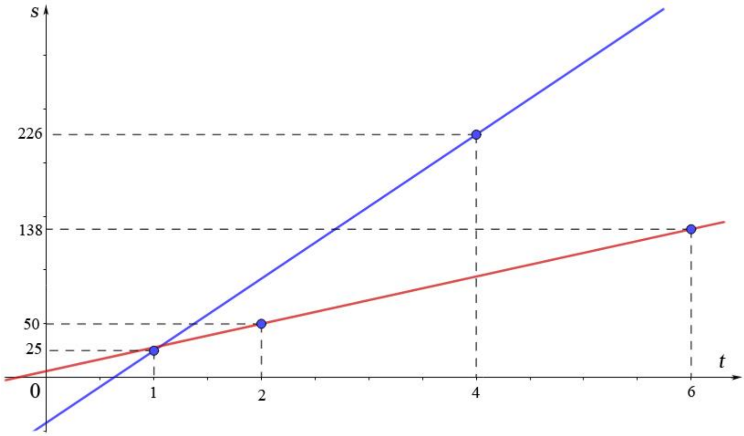


Figure 3.3

The slope of the line is the rate of change of the dependent variable from the argument, in this case, the distance from time. That is, in this case, the slope of the first straight line is the speed of the motorcycle: $\frac{226-25}{4-1}=\frac{201}{3}=67$;

second straight line — cyclist: $\frac{138-50}{6-2}=\frac{88}{4}=22.$

It can be seen that a higher speed is expressed by a large slope of a straight line.

Comment

It is important to note that when we talk about speed, we mean an average speed.

Exercises 3.3

C. At a price of 80 soms, 25 tons of rice are offered on the market, at a price of 100 soms there are 30 tons. Draw a straight line passing through these points and determine the slope of the straight line of the supply for rice on this market, mark the amount of rice on the horizontal axis, and the price on the vertical axis.

At a price of 87 soms, 32 tons of rice to be purchased on the market, at a price of 82 soms there are 36 tons. Determine the slope of the line of the demand for rice in this market. Draw its graph in the same coordinate plane as the supply graph.

H. At a price of 150 soms 2 tons of lemons are offered on the market, at a price of 125 soms there are 1.6 tons. Draw a line which passes through these points and determine the slope of the line of the supply for lemons in this market, mark the number of lemons on the horizontal axis, and the price on the vertical axis.

At a price of 165 soms, costumers are ready to buy 1.42 tons of lemons in the market, at a price of 143 KGS there are 1.97 tons. Determine the slope of the line of the demand for lemons in this the market. Draw its graph in the same coordinate plane as the supply graph.

3.4. Problem

A car drove from Naryn to Bishkek and moved at a speed of 80 km / h. The distance between Naryn and Bishkek is 315 kilometers. At what distance from Bishkek he was in

a) 1 hour;

b) 2.5 hours;

c) 2 hours 54 minutes;

d) 3 hours and 15 minutes?

Draw a Cartesian coordinate system on the plane and mark the corresponding points on it. Make sure that you can draw a straight line through these points.

Solution

At the initial moment of time, that is, at $t = 0$, the distance is 315, then every hour it will be reduced by 80 km. Therefore, the distance will be determined by the formula $s = 315 – 80t$, where t is the time measured in hours.

Substituting the time values into the formula, we will receive answers in (km):

a) $s = 315 – 80·1 = 235$;

b) $s = 315 – 80·2.5 = 115$;

c) $s = 315 – 80·2.9 = 83$;

d) $s = 315 – 80·3.25 = 55$.

In this case, on the coordinate plane should be marked points: *(1, 235), (2.5, 115), (2.9, 83), (3.25, 55).*

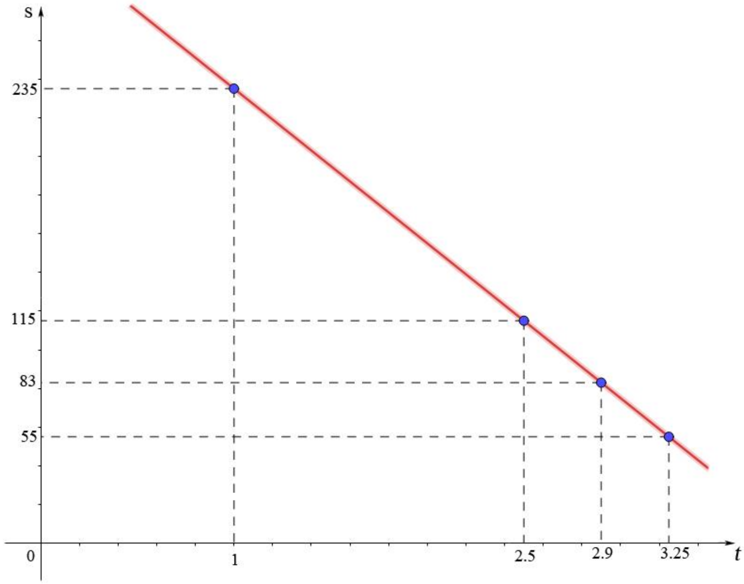


Figure 3.4

With a ruler, make sure that all these points lie on the same straight line.

Exercises 3.4

C. Diana has 300 soms and buys grapes at a price of 80 soms per kg. How much money will remain after purchase

a) 2 kg; b) 0.5 kg of grapes?

How many kilograms of grapes Diana bought if she has

c)100 soms d) 200 soms left?

Draw a Cartesian coordinate system on the plane and mark the corresponding points on it, mark the weight of the grapes on the horizontal axis and the number of soms on the vertical axis. Make sure you can draw a straight line through these points. Write a function of the amount of soms which depends on the weight of the grapes purchased.

H. In the barrel 80 liters of water, which spills out with a speed of 4 liters per minute. How many liters of water will be in the barrel after

a) 5 minutes; b) 15 minutes?

After how many minutes in the barrel will remain

c) 52 liters ; d) 40 liters of water?

Draw a Cartesian coordinate system on the plane and mark the corresponding points on it, mark the number of minutes on the horizontal axis and the number of liters of water on the vertical axis. Make sure you can draw a straight line through these points. Write a function of the volume of water in the barrel which depends on time in minutes.

3.5. Problem

At 2:24 PM the motorcycle was at a distance of 150 km from Bishkek and was moving towards Naryn at a speed of 56 km / hour. Write the appropriate function and determine a distance between motorcycle and Bishkek at 4:00 PM.

Check the answer.

Solution

As already mentioned, the movement at a constant speed can be described by the function $y = mx + b$, where x will be the time, and y is the distance. The slope of the line is given by the speed — that is, the function $y = 56x + b$ holds.

In order to find the value of the coefficient b, we use the condition "At 2:24 PM the motorcycle was at a distance of

150 km from Bishkek”. Since 2 hours and 24 minutes is 2.4 hours, we get the equation $150=56\cdot2.4 + b$. Hence, $b=15.6$. So, the movement of the motorcycle is described by the function $y = 56x + 15.6$ and at 4 o'clock it was at a distance: $y = 56 \cdot 4 + 15.6 = 239.6$ kilometers from Bishkek.
Check the answer is easy. Since for: $4 – 2.4 = 1.6$ hours the motorcyclist will cover $56 \cdot 1.6 = 89.6$ kilometers, he will be at a distance: $89.6 + 150 = 239.6$ kilometers from Bishkek.

So, we have shown that the equation of a straight line can be determined unambiguously if the slope and the coordinates of the point lying on this straight line are known.

Exercises 3.5

C. Camron sells apricots at a price of 45 soms. After selling 4 kg, he had 250 soms in his pocket. Write a function of the amount of soms which depends on the weight of apricots sold. How many soms will he have after selling 6.4 kg of apricots?

H. Water from the barrel is spilling out with a speed of 8 liters per minute. After 5 minutes 54 liters left in the barrel. Write a function of the volume of water in the barrel which depends on time in minutes. How much water will remain in the barrel after 8.5 minutes?

3.6. Problem

Write the equation of a straight line that is parallel to the straight line k and passes through the point M.

a) Straight line k: $y = 2x – 1$;   M (2, 7);

b) Straight line k: $y = – 0.5x + 6$;   M (2, 3).

Draw the initial and the resulting straight lines.

Solution

As already mentioned, parallel lines have the same slope. Therefore, it is sufficient to determine the value of the free coefficient.

a) The resulting equation: $y = 2x + b$. Substituting the coordinates of the point M, we get $7 = 2 \cdot 2 + b$.
Thus, $y = 2x + 3$.
To draw a straight line, it is enough to have two points. As one of them you can always use the point of intersection with the vertical axis — the value of the coefficient b. The coordinates of the second point in the resulting equation are known from the condition, and in the initial straight line, you can choose some value of the argument and calculate the corresponding value of the function using the equation. For example, if $x = 2$, then $y = 3$.

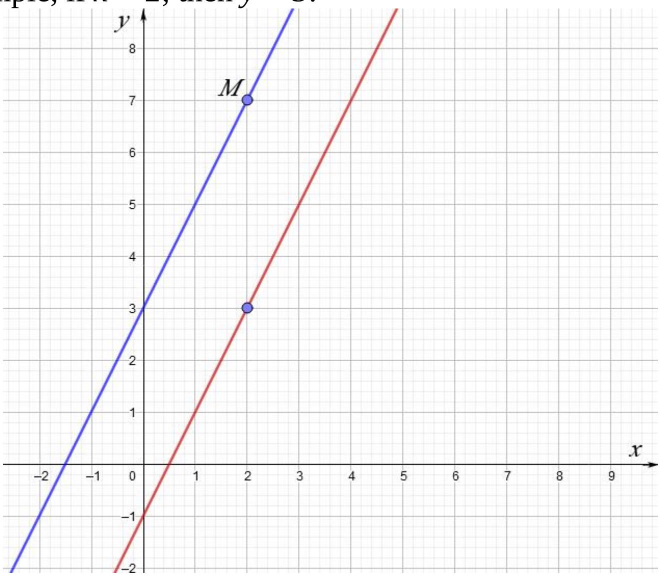


Figure 3.5

b) The resulting equation: $y = – 0.5x + b$. Substituting the coordinates of the point M, we get $3 = – 0.5 \cdot 2 + b$. Thus, $y = – 0.5x + 4$. In order to find the coordinates of the second point of the original line, you can take $x = 12$, then $y = 0$.

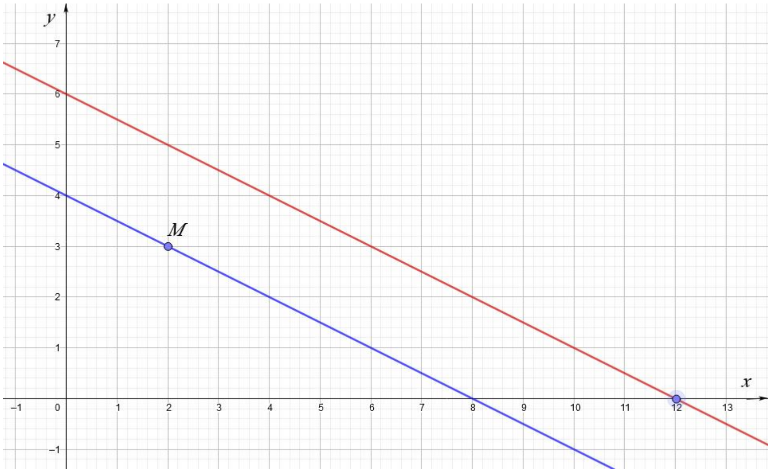


Figure 3.6

We have already said that in order to draw a straight line, it is enough to have two points. Consequently, in order to write an equation of a straight line, it suffices to know the coordinates of two points lying on this straight line.

Exercises 3.6

C. Write the equation of a straight line that is parallel to the straight line *k* and passes through the point *M*.

a) Straight line *k*: $y = 4x + 2$; $M(2, 6)$;

b) Straight line *k*: $y = -0.2x - 1$; $M(5, 3)$.

H. Write the equation of a straight line that is parallel to the straight line *k* and passes through the point *M*.

a) Straight line *k*: $y = -2x - 1$; $M(2, -1)$;

b) Straight line *k*: $y = 3x + 6$; $M(2, 3)$.

3.7. Problem

At 1:36 PM the cyclist was at a distance of 50 km from Bishkek, and at 5:00 PM he was at a distance of 135 km from Bishkek. At what distance from Bishkek, he was at 3:30 PM? What time was he at a distance of 130 km from Bishkek at? It is assumed that he was moving with a constant speed.

Solution

Motion with a constant speed is described by a function of the form $y = mx + b$, where x will express time and y the distance. Since the units of measurement are hours and kilometers, we note that 1 hour 36 minutes is 1.6 hours, and 3 hours 30 minutes is 3.5 hours.

Substituting in the function $y = mx + b$ the coordinates of two given points, we obtain a system of two linear equations of the first order: $\begin{cases} 50 = m\cdot 1.6 + b; \\ 135 = m\cdot 5 + b; \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 50 = m\cdot 1.6 + b; \\ 85 = m\cdot 3.4; \end{cases} \Leftrightarrow \begin{cases} b = 50 - 25\cdot 1.6 = 10; \\ m = 25. \end{cases}$$

We found that the movement of a cyclist is described by the function $y = 25x + 10$, where 25 is its speed in km / h. From here it is easy to find out that at 3:30 PM he was at a distance: $y = 25 \cdot 3.5 + 10 = 97.5$ kilometers from Bishkek, and at a distance of 130 km from Bishkek he was at: $130 = 25x + 10 \Leftrightarrow x = 4.8$ hours.

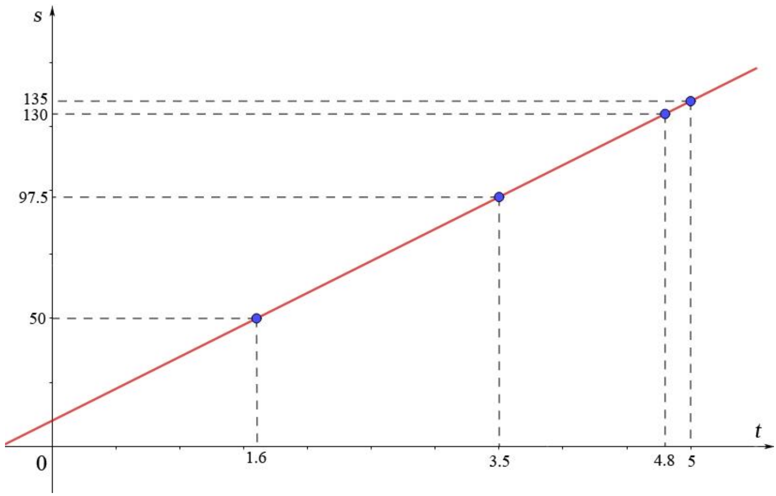


Figure 3.7

Exercises 3.7

C. At a price of 70 soms, 25 tons of bananas are offered on the market, at a price of 90 soms there are 30 tons. Draw a line through these points. Write the banana supply linear function as the price function of quantity.

At a price of 77 soms, 32 tons of bananas are offered on the market, at a price of 72 soms there are 36 tons. Draw a line through these points. Write the linear demand function for bananas as the price function of quantity.

Determine the equilibrium price and equilibrium quantity for this market.

H. At a price of 160 soms, 2 tons of cookies are offered on the market, at a price of 135 soms there are 1.6 tons. Draw a line through these points. Write the cookies supply linear function as the price function of quantity.

At a price of 175 soms, 1.42 tons of cookies are offered on the market, at a price of 153 soms there are 1.97 tons. Draw a line through these points. Write the demand linear function for cookies as the price function of quantity.

Determine the equilibrium price and equilibrium quantity for this market.

3.8. Problem

A train left the city A at 1 :00 PM and arrived in the city of B at 7 :00 PM. At 10:30 PM, another train, which left city B at 1:30 PM, arrived at A. Determine when and at what distance from A met the train, if the distance between these cities is 360 kilometers.

Solution

In the problems with similar content, by default, motion is assumed to be at a constant rate. Therefore, it can be said that relation between the distance traveled and time is described by a linear function $y = mx + b$.

In this case, it is more convenient to write this function in the form $s = vt + s_0$. Here, s is the distance; v is the speed; t is the time; s_0 is the "location of the object at the initial time".

Consider A as the origin point. Then, on the coordinate plane (t, s) the movement of the first train is described by the straight line EL, defined by the points $E(1, 0)$ and $L(7, 360)$, and the displacement of the second train is described by the straight line SR, where coordinates the points: $S(1.5, 360)$ and $R(10.5, 0)$.

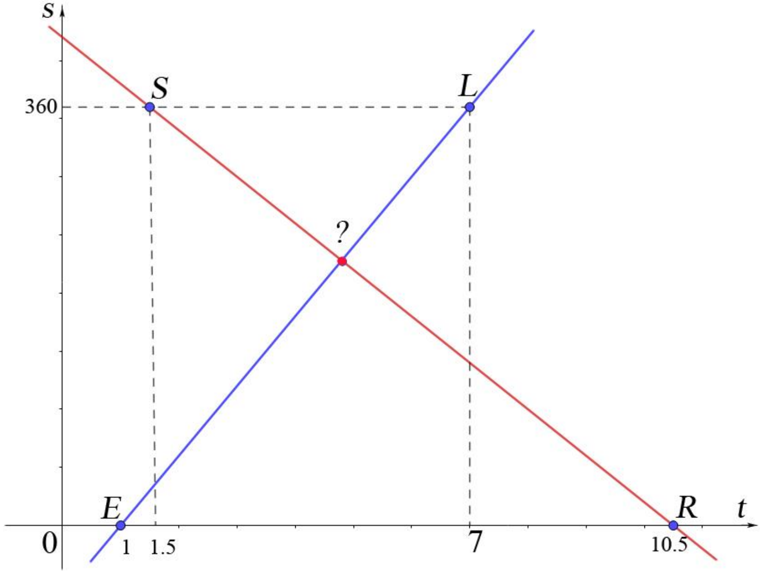


Figure 3.8

In order to obtain the equation of the line EL in the form $s = vt + s_0$, we use the coordinates of the points E and L:

$$\begin{cases} 0 = v \cdot 1 + s_0, \\ 360 = v \cdot 7 + s_0, \end{cases} \Leftrightarrow \begin{cases} 0 = v \cdot 1 + s_0, \\ 360 = v \cdot 6, \end{cases} \Leftrightarrow \begin{cases} s_0 = -v \cdot 1 = -60, \\ v = 60. \end{cases}$$

So, it turned out that the first train was moving at a speed of 60 km/h and its motion is described by the equation $s = 60t - 60$.

In order to obtain the equation of the straight line *SR* describing the displacement of the second train, we use the coordinates of the corresponding points:

$$\begin{cases} 360 = v \cdot 1.5 + s_0, \\ 0 = v \cdot 10.5 + s_0, \end{cases} \Leftrightarrow \begin{cases} 360 = v \cdot 1.5 + s_0, \\ 360 = -9v, \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} s_0 = 360 - v \cdot 1.5 = 420, \\ v = -40. \end{cases}$$

Thus, the distance on which the second train is far from *A* is determined by the equation $s = -40t + 420$. A minus sign before the speed value indicates that this train is moving in the opposite direction to the direction of the first train.

In order to determine the coordinates of the meeting point of trains it remains to solve the corresponding system:

$$\begin{cases} s = 60t - 60, \\ s = -40t + 420, \end{cases} \Leftrightarrow \begin{cases} s = 60t - 60, \\ 60t - 60 = -40t + 420, \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} s = 60t - 60, \\ 100t = 480, \end{cases} \Leftrightarrow \begin{cases} s = 60 \cdot 4.8 - 60 = 228, \\ t = 4.8. \end{cases}$$

It turned out that they met at 4.8 hours, that is at 4:48 PM at a distance of 228 km from *A*.

Exercises 3.8

C. Dinara began weeding the lawn at 1:20 PM. At 1:30 PM from the other side of the lawn, Mukhtar set to work. What time will they finish weeding this lawn? It is known that Dinara can weed such lawn in 1 hour, and Mukhtar in 40 minutes.

H. A bus left Bishkek at 3:00 PM and after breaking 180 km arrived in Balykchi at 7:00 PM. At 4:36 PM, the bus met a car that left Balykchi at 3:15 PM. At what distance did they meet from Bishkek? What time did the car arrive in Bishkek?

Summary

1. The seamstress sewed 14 shirts in 2 days, and 35 shirts in 5 days. How many shirts were sewn in 4 days? In how many days, were 49 shirts sewn? Constant labor productivity is assumed. Draw a Cartesian coordinate system on the plane and mark the corresponding points on it, mark the time on the horizontal axis, and the number of shirts on the vertical axis. Make sure you can draw a straight line through these points.
2. At 1:00 PM in the afternoon there were 8 tons of flour in the warehouse, at 3:30 PM there were 4 tons. Write the equation of the dependence of the amount of flour on time, assuming a linear relationship. Draw the appropriate line. How much flour was in stock at 4:15 PM? What time was 6.08 tons in stock?
3. Write the equation of a straight line that is parallel to the straight line k and passes through the point M.
a) Straight line k: $y = -x + 3;\ M(9, -9);$
b) Straight line k: $y = 0.3x - 2;\ M(10, 7)$.
4. At a price of 40 soms, 45 tons of apples are offered on the market, at a price of 50 soms there are 53 tons. Draw a line through these points. Write a linear supply function as the price function of quantity.
At a price of 47 soms in the market, consumers are ready to buy 52 tons of apples, at a price of 52 soms there are 48 tons. Draw a line through these points. Write a linear demand function as the price function of quantity.
Determine the equilibrium price and equilibrium quantity for this market.
5. At a price of 254.5 soms, 1.5 tons of sweets are offered on the market, at a price of 279.5 soms there are 1.7 tons. Draw a line through these points. Write a linear supply function as the price function of quantity.

At a price of 275 soms, consumers are ready to buy 1.4 tons of sweets in the market, at a price of 255 soms there are 1.9 tons. Draw a line through these points. Write a linear demand function as the price function of quantity.
Determine the equilibrium price and equilibrium quantity for this market.

6. At 2:24 PM the motorcycle was at a distance of 150 km from Bishkek and was moving towards Bishkek at a speed of 56 km / h. Write the corresponding function and determine at what distance the motorcycle was from Bishkek at 4:00 PM. Check the answer.

7. A bus left the city of Naryn at 1:36 PM and arrived in Bishkek at 8:00 PM. On the same day at 2:18 PM a taxi left Bishkek and arrived in Naryn at 7:18 PM. Given they were moving at constant speeds, determine when they met.

8. From Lukashovka and Mokrousovka, the distance between which is equal to 16 kilometers, Ira and Kolya came out to meet each other. Ira left at 1:45 PM and arrived at Mokrousovka at 5:45 PM. Kolya hit the road at 3:00 PM and arrived at Lukashovka at 6:20 PM. Determine when and at what distance from Lukashovka they will meet.

9. Zhanara started weeding the lawn at 1:55 PM. At 2:00 PM, the same lawn from the other end began to weed Nasip. What time will they finish weeding this lawn? It is known that Zhanara can weed such a lawn alone in 25 minutes, and Nasip in 15 minutes.

§4. Intro to Cost–Volume–Profit – CVP Analysis

Economic resources is one of the basic concepts of economic theory. Adam Smith considered economic resources such as labor, land, and capital. However, the theory of the three factors of production was most clearly formulated by the French economist Jean Baptiste Say (1767–1832). The English economist Alfred Marshall (1842–1924) proposed adding a fourth factor — entrepreneurial ability. The argument of the supporters of these approaches continued until the end of the twentieth century. The collapse of the Soviet Union and the collapse of the post–Soviet economies clearly demonstrated the importance of entrepreneurial ability. These countries had labor, land and capital, but the actual lack of entrepreneurial abilities led to a sharp reduction in GDP. So, by the year 2000, 10 years of the transition from a command economy to a market economy, Latvia lost 36% of GDP, Lithuania –35% of GDP, the Russian Federation – 38% of GDP, Ukraine – 58% of GDP.

At the same time, it is important to note that entrepreneurial skills need to be supported by appropriate knowledge. For example, in the USA about 600 thousand small enterprises are registered annually and about 500 thousand are liquidated. Of course, there are many different reasons explaining the number of liquidations, but the lack of relevant knowledge is among the determining ones.

This paper proposes a simple and visual introduction to the CVP model (Cost – Volume – Profit), which explains in an accessible way the process of finding a break point – BEP (break-even point). Despite the seeming simplicity, the correct definition of BEP allows you to succeed in business, and mistakes can cause huge losses.

We illustrate this statement with the following media report: Airbus ceases production of A380 wide-body aircraft. In 2021, he will complete their delivery. The reason is the company has not reached the BEP. Airbus initially estimated the break-even point at 420 units, but by the end of January 2019, the A380 had only 316 orders. The project absorbed 28 billion euros, which seems unrecoverable.

4.1. Problem

Sinbad the Sailor, who carries chocolate to the ASINAM island, sells a box of chocolate for 12 dirhams[1]*. What is the revenue of Sinbad the Sailor who sold*
a) 5 b) 12.5 c) 30 d) 40 boxes of chocolate?

Draw a Cartesian coordinate system on the plane and mark the corresponding points on it. Make sure that you can draw a straight line through these points.

Solution

Revenue is determined by the formula $R = 12q$, where q is number of boxes sold. Substituting into the formula the number of boxes, we will receive answers in dirhams:

a) $R(5) = 12 \cdot 5 = 60$;
b) $R(12.5) = 12 \cdot 12.5 = 150$;
c) $R(30) = 12 \cdot 30 = 360$;
d) $R(40) = 12 \cdot 40 = 480$.

Thus, on the coordinate plane you need to mark the points: $(5, 60)$, $(12.5, 150)$, $(30, 360)$, $(40, 480)$, where the first coordinates indicate the number of boxes and are marked on the horizontal axis $0q$, and the second coordinates show

[1]Dirham is an old Arabic coin.

the amount of revenue of Sinbad the Sailor and are indicated on the vertical axis *0R*.

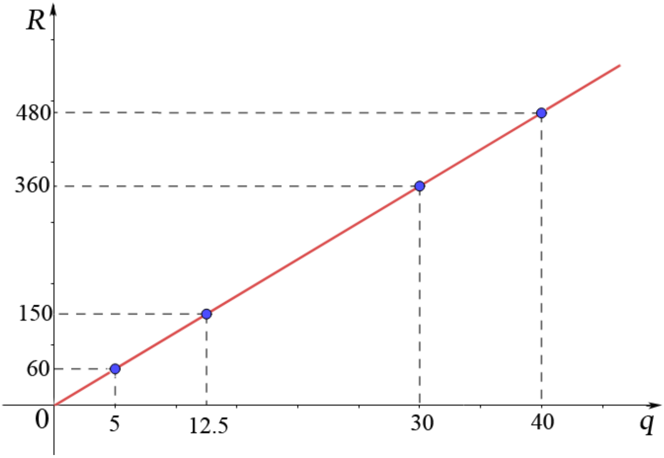


Figure 4.1

With a ruler, make sure that all these points lie on a straight line.

Exercises 4.1

C. Aidana sells T-shirts with the AUCA logo for the price of 200 soms. What is the revenue of Aidana who sold

a) 25 b) 30 c) 40 T-shirts?

Draw a Cartesian coordinate system on the plane, placing the number of T-shirts horizontally, and mark the corresponding points on it. Make sure that you can draw a straight line through these points.

H. Nilufar sells dried apricots at a price of $4 per kilogram. What is the revenue of Nilufar, who sold

a) 50 b) 65 c) 70 kilograms of apricots?

Draw a Cartesian coordinate system on the plane, placing the kilograms horizontally, and mark the corresponding points on it. Make sure that you can draw a straight line through these points.

4.2. Problem

Sinbad the Sailor buys on the mainland each box of chocolate for 8 dirhams. What are the costs of Sinbad the Sailor to buy
a) 15 b) 25 c) 30 d) 40 boxes of chocolate?

Draw a Cartesian coordinate system on the plane and mark the corresponding points on it. Make sure that you can draw a straight line through these points.

Solution

The cost of buying the appropriate number of boxes of chocolate is generally called variable cost (VC). In this case, they are determined by the formula $VC = 8q$, where q is the number of bought boxes.

Substituting into the formula the number of boxes, we will receive answers in dirhams:

a) $VC(15) = 8 \cdot 15 = 120$;
b) $VC(25) = 8 \cdot 25 = 200$;
c) $VC(30) = 8 \cdot 30 = 240$;
d) $VC(40) = 8 \cdot 40 = 320$.

Thus, on the coordinate plane you need to mark the points: $(15, 120)$, $(25, 200)$, $(30, 240)$, $(40, 320)$, where the first coordinates indicate the number of boxes and are marked on the horizontal axis $0q$, and the second coordinates show how many dirhams Sinbad the Sailor paid for the corresponding number of boxes of chocolate and are indicated on the vertical axis $0VC$.

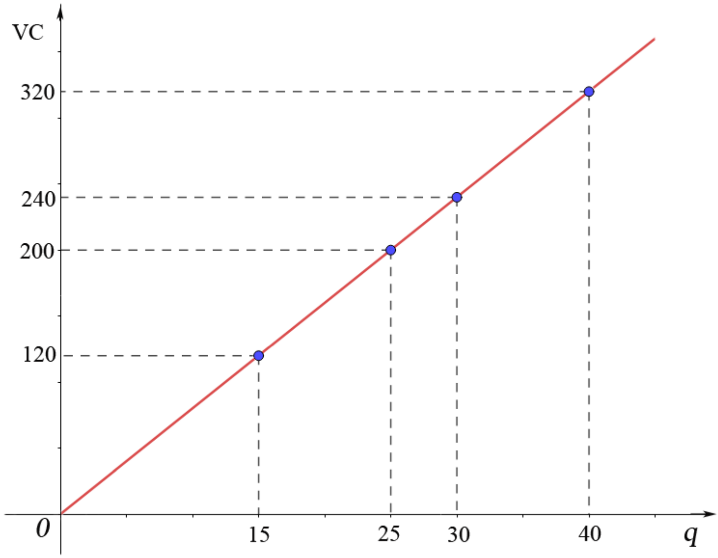


Figure 4.2

With a ruler, make sure that all these points lie on a straight line.

Exercises 4.2

C. Aidana buys plain T-shirts at the price of 100 soms in order to put on them the AUCA logo. In addition to the job of applying the emblem: payment to workers, paint, equipment etc., 60 soms for each T-shirt are required. What are the variable costs of Aidana, who made

a) 20 b) 35 c) 45 T-shirts with an emblem?

Draw a Cartesian coordinate system on the plane, placing the number of T–shirts horizontally, and mark the corresponding points on it. Make sure you can draw a straight line through these points.

H. Nilufar buys apricots at a price of $ 1.4 per kilogram. In addition, $ 0.6 per kilogram is spent on apricot drying work. What are the variable costs of Nilufar, who prepared for sale

a) 450 b) 600 c) 900 kilograms of dried apricots?

Draw a Cartesian coordinate system on the plane, placing the kilograms horizontally, and mark the corresponding points on it. Make sure you can draw a straight line through these points.

4.3. Problem

Sinbad the Sailor buys every box of chocolate on the mainland for 8 dirhams. The fixed costs for each voyage (sailor's salary, food, taxes to sultan, etc.) are 140 dirhams. What is the total cost of Sinbad the Sailor who bought
a) 10 b) 20 c) 37.5 d) 45 boxes of chocolate?
Mark the corresponding points in Figure 4.2. Make sure that you can draw a straight line through these points.
In the same figure, draw a line expressing fixed costs. What can be figured out as a result of the joint consideration of the three lines?

Solution

Total costs(TC) are the sum of variable cost(VC) and fixed cost(FC). Therefore, the total cost of Sinbad the Sailor in this case is determined by the formula $TC = VC + FC = 8q + 140$, where q is the number of bought boxes. Substituting into the formula the number of boxes, we will receive answers in dirhams:

a) $TC(10) = 8\cdot 10 + 140 = 220$;
b) $TC(20) = 8\cdot 20 + 140 = 300$;
c) $TC(37.5) = 8\cdot 37.5 + 140 = 440$;
d) $TC(45) = 8\cdot 45 + 140 = 500$.

Thus, in Figure 4.2 you need to mark the points: $(10, 220)$, $(20, 300)$, $(37.5, 440)$, $(45, 500)$, where the first coordinates indicate the number of boxes and are marked on the horizontal axis $0q$, and the second coordinates show how many dirhams Sinbad the Sailor spent and are marked on the vertical axis $0TC$.

With a ruler, make sure that all these points lie on a straight line.

Because any number of boxes of chocolate that Sinbad the Sailor bought and took to float corresponds to the same value of fixed costs, the schedule of fixed costs function is a horizontal line *FC = 140.*

It should be noted that although in problems 4.2 and 4.3 on the vertical axis are marked: variable costs, total costs, fixed costs, they are all expressed in dirhams (*Dh*) and therefore can be marked on the same axis in Figure 4.3.

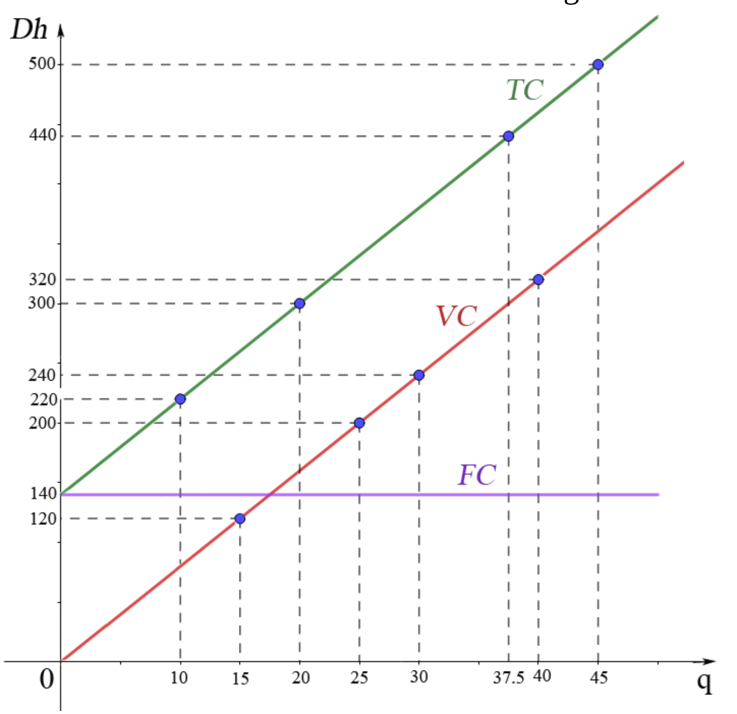


Figure 4.3

It is easy to understand that a straight line expressing total costs is obtained by a parallel shift of a straight line expressing variable costs by the amount of fixed costs.

Exercises 4.3

C. In the conditions of exercise 4.2.C, Aidana's fixed costs: rental of premises, equipment, salary of employees etc., are 5000 soms. What are the total costs of Aidana, who made

a) 80 b) 135 c) 150 T - shirts with an emblem?

Mark the corresponding points at figure for exercise 4.2.C. Make sure you can draw a straight line through these points.

H. In the conditions of exercise 4.2.H fixed costs of Nilufar: rental of premises, equipment, salary of employees etc., equal to $1300. What are the total costs of Nilufar, who prepared for sale

a) 400 b) 550 c) 800 kilograms of dried apricots?

Mark the corresponding points at figure for exercise 4.2.H. Make sure you can draw a straight line through these points.

4.4. Problem

Sinbad the Sailor, who came to the market to buy chocolate had 400 dirhams. He bought each box at the price of 8 dirhams. How many dirhams did he have after buying

a) 10 b) 22.5 c) 32.5 d) 40 boxes of chocolate?

Draw a Cartesian coordinate system on the plane and mark the corresponding points on it. Make sure these points lie on the same straight line.

Solution

Denote by M the amount of money from Sinbad the Sailor. Then $M = 400 – 8q$, where q is the number of bought boxes. Substituting into the formula the number of boxes, we will receive answers in dirhams:

a) $M(10) = 400 – 8·10 = 320$;

b) $M(22.5) = 400 – 8·22.5 + 140 = 220$;

c) $M(32.5) = 400 – 8·32.5 + 140 = 140$;

d) $M(40) = 400 – 8·40 + 140 = 80$.

On the coordinate plane you need to mark the points: $(10, 320)$, $(22.5, 220)$, $(32.5, 140)$, $(40, 80)$, and with a ruler, make sure that in this case we get points lying on one straight line. In contrast to the previous cases, the straight line is directed to the right-down.

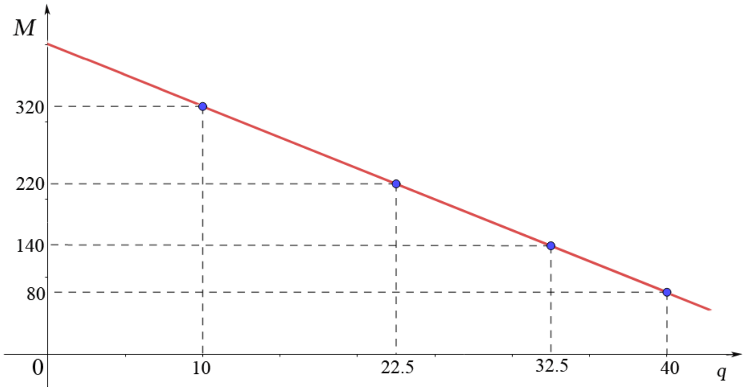


Figure 4.4

Remark

Functions $R = 12q$, $VC = 8q$, $TC = 8q + 140$, $M = 400 – 8q$ are special cases of a linear function, which is usually written as $y = mx + b$ and is called the equation of a straight line.

The number b is the coordinate of the point of intersection of the line with the vertical axis(***y* – intercept**).

The coefficient m is called the **slope** (angular coefficient) of a straight line.

It is easy to understand that straight lines with the same slope – the same coefficient m are parallel.

Exercises 4.4

C. Aidana, who buys plain T-shirts for 100 soms at the market, had 19 000 soms. How much money Aidana left after the purchase

a) 180 b) 125 c) 90 plain T-shirts?

In a Cartesian coordinate system on a plane, mark the corresponding points. Make sure you can draw a straight line through these points.

How will the line change, if we assume that Aidana had not 19 000, but 20 500 soms.

H. Nilufar has $1 530. How many dollars will remain with her after the purchase

a) 450 b) 600 c) 1000 kilograms of apricots for $1.4? In a Cartesian coordinate system on a plane, mark the corresponding points. Make sure you can draw a straight line through these points. How will the line change, if we assume that Nilufar had not $1530, but $1450.

4.5. Problem

Sinbad the Sailor carries chocolate to the ASINAM island and sells a box of chocolate for 12 dirhams. Each box of chocolate he buys on the mainland for 8 dirhams. The fixed costs for each voyage (sailor's salary, food, taxes to sultan, etc.) are 140 dirhams.

What is the profit of Sinbad the Sailor from the sale on the island of all

a) 15 b) 30 c) 37.5 d) 45 boxes of chocolate?

Solution

Profit ***Pf*** is the difference between revenue and total costs: ***Pf = R – TC***. It can be calculated in two methods.

The first method: Calculate the values of revenue and total costs with the appropriate number of boxes, and then find their difference. Let's make calculations in the form of a table.

Number of boxes ***q***	*15*	*30*	*37.5*	*45*
Revenue ***R = 12q***	*180*	*360*	*450*	*540*
Total costs ***TC = 8q + 140***	*260*	*380*	*440*	*500*
Profit ***Pf = R – TC***	*– 80*	*– 20*	*10*	*40*

The second method: Find the profit function as the difference between the revenue function and total costs: $Pf = R - TC = 12q - (8q + 140) = 4q - 140$, and then find its values at the points:

a) $Pf(15) = 4 \cdot 15 - 140 = -80$;

b) $Pf(30) = 4 \cdot 30 - 140 = -20$;

c) $Pf(37.5) = 4 \cdot 37.5 - 140 = 10$;

d) $Pf(45) = 4 \cdot 45 - 140 = 40$.

Note that at the points $q = 15$ and $q = 30$ the profit value is negative — there are losses, and at points $q = 37.5$ and $q = 45$ the profit value is positive. Of course, it would be good to know the value of q that separates the areas of the negative and positive profit. This is the next problem.

Exercises 4.5

C. Aidana sells T-shirts with the AUCA logo for the price of 200 soms. Its average variable costs for making such a T-shirt are 160 soms, fixed costs 5 000 soms. What is the profit of Aidana, who made and sold

a) 95 b) 130 c) 150 T-shirts?

Draw a Cartesian coordinate system on the plane, placing the number of T-shirts horizontally, and mark the corresponding points on it. Make sure you can draw a straight line through these points.

H. Nilufar sells dried apricots at a price of $4 per kilogram. Its variable costs per kilogram of apricots are $2, fixed costs $1300. What is the profit of Nilufar, who dried and sold

a) 550 b) 600 c) 800 kilograms of apricots?

Draw a Cartesian coordinate system on the plane, placing the kilograms horizontally, and mark the corresponding points on it. Make sure you can draw a straight line through these points.

4.6. Problem

Sinbad the Sailor carries chocolate to the ASINAM island. The fixed costs for each voyage (sailor's salary, food, taxes to sultan, etc.) are 140 dirhams.

How many boxes of chocolate should I buy on the mainland for 8 dirhams, and then sell Sinbad the Sailor on *the Asinam island for 12 dirhams in order to fully recoup all his expenses?*

Solution

There are two methods to solving this problem.

The first method: Sinbad the Sailor will fully recoup all his costs if his revenue equals the total costs. That is, the equality $\boldsymbol{R = TC}$ should hold. So, $12q = 8q + 140$. From here, $q = 35$. Thus it turned out that Sinbad the Sailor will fully pay back all his expenses if he buys on the mainland on the island and then sells 35 boxes of chocolate on the island

Very useful to draw a graph. In the same coordinate system, it is necessary to draw graphs of revenue functions and total costs that define the left and right sides of the equation $12q = 8q + 140$.

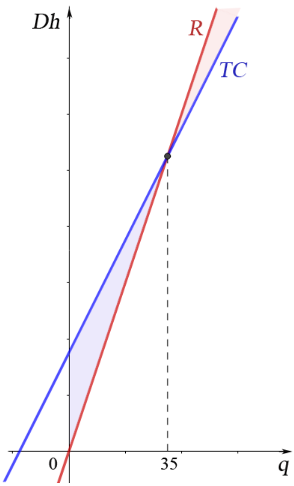


Figure 4.5

It is known that the root of the equation is the coordinate of the point of intersection of the graphs. In economics, the first coordinate of the point of intersection of revenue schedules and total costs is called the **break – even point**. She divides the axis [0; ∞) into a loss zone and a profit zone.

The second method: We write the profit function as the difference between the functions of revenue and total costs: $Pf = R - TC = 12q - (8q + 140) = 4q - 140$, and then equate it to zero: $4q - 140 = 0$. The root of this equation determines the break – even point.

Draw a graph of the profit function $Pf = 4q - 140$. Recall that any straight line is defined by two points – you need to mark two points, with a ruler and draw a straight line. As one point of the line $y = mx + b$, it is convenient to take the value of the coefficient b, and the second point can be obtained by substituting some value for the argument x. In our case, $b = -140$, and the second point is obtained at $q = 40$: $Pf(40) = 4 \cdot 40 - 140 = 20$.

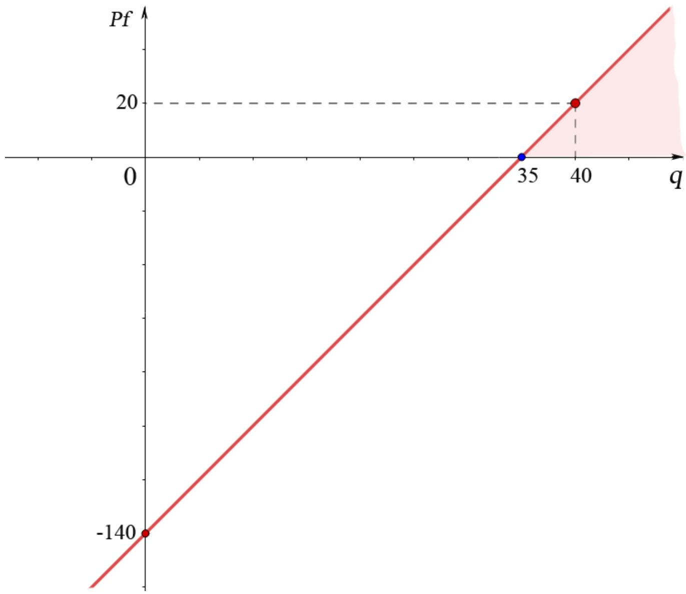


Figure 4.6

This very simple model from the point of view of mathematics is very important from the point of view of the business. Sinbad the Sailor, before he starts buying chocolate and hiring a crew of sailors, must at least approximately find a break-even point. He can start a business only if he is sure that he can sell more chocolate than the quantity at the break-even point.

Exercises 4.6

C. In the conditions of exercise 4.5.C, determine how many T-shirts Aidana has to make and sell in order to recoup all her expenses?

H. In the conditions of exercise 4.5.H, determine how many kilograms of apricots Nilufar must dry and sell in order to recoup all her costs?

4.7. Problem

The Sultan, in whose possessions Sinbad the Sailor works, needed money to buy jewelry for his beloved wife. Therefore, he raised taxes.

As a result, fixed costs for each voyage increased to 170 dirhams. The remaining data has not changed: Sinbad the Sailor buys every box of chocolate on the mainland for 8 dirhams and sells on the ASINAM island for 12 dirhams. How many boxes of chocolate Sinbad the Sailor bought and sold during the next voyage, if

a) he returned the money spent;

b) he made a profit of 40 dirhams;

c) his losses amounted to 22 dirham?

Solution

Compared to the previous situation will change only the total cost function: instead of $TC = 8q + 140$ will $TC_1 = 8q + 170$. It is convenient to use the second approach of the previous problem.

We write the profit function as the difference between the proceeds and the total cost function: $Pf = R - TC_1 = 12q - (8q + 170) = 4q - 170$, a and then to equate it to the desired value. As a result, we get three equations:

a) he returned the money spent ⇔ $4q – 170 = 0$;

b) he made a profit of 40 dirhams ⇔ $4q – 170 = 40$;

c) his losses amounted to 22 dirham ⇔ $4q – 170 = – 22$.

Solving these equations, we find that

a) he returned the money spent if he bought and sold 42.5 boxes of chocolate;

b) he made a profit equal to 40 dirhams if he bought and sold 52.5 boxes of chocolate;

c) his losses amounted to 22 dirhams if he managed to buy and sell only 37 boxes of chocolate.

We draw graphs of the profit functions and see that a change in the free term, in this case 140 by 170, leads to a parallel shift of the straight line. From the point of view of economy we get the obvious thing that an increase in the cost increases loss zone and reduces the profit zone.

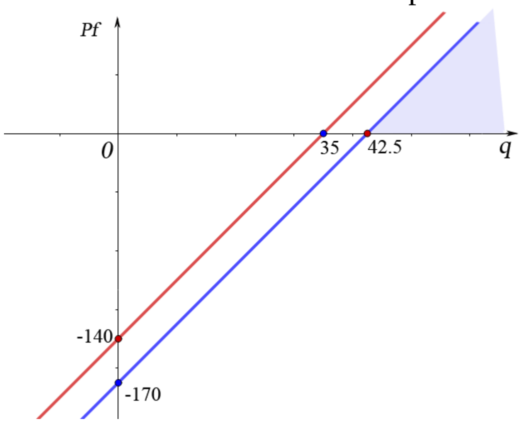


Figure 4.7

Exercises 4.7

C. In the conditions of exercise 4.5.C, the owner of the premises increased the rent by 600 soms. Determine how many T-shirts Aidana made and sold if

a) she paid back all her expenses;

b) made a profit of 1000 soms;
c) her losses amounted to 240 soms.
H. In the conditions of exercise 4.5.H, the cost of equipment rental increased by $44. Determine how many kilograms of apricots dried and sold by Nilufar, if
a) she paid back all her expenses;
b) made a profit of $100;
c) her losses amounted to $24.

4.8. Problem

When it became known about buying jewelry for his beloved wife, the other wives of the Sultan staged a protest, as a result of which the Sultan promised to buy them jewelry, once again raising taxes.

As a result, now the cost of each voyage (fixed costs) of Sinbad the Sailor is 180 dirhams. In addition, as a result of the tax increase, chocolate has also risen in price. At what price for a box did Sinbad the Sailor buy 50 boxes of chocolate if his total expenses were 600 dirhams?

Solution

As already indicated, the total costs are determined by the formula $TC = FC + VC$, where the magnitude of the variable costs is determined by the formula $VC = AVC \cdot q$. Here AVC is the unit cost, q is the number of goods bought.

So, from the conditions of the problem we get: $600 = 180 + AVC \cdot 50$. Thus it turns out that Sinbad the Sailor paid for each box of chocolate: $(600 – 180)/50 = 8.4$ dirhams.

Exercises 4.8

C. As a result of the introduction of the new tax, Aidana's fixed costs increased by 800 soms. Variable costs have also risen. What now are the average variable costs of making T-shirts with the AUCA emblem equal if the total cost of making 120 T-shirts is 26 440 soms?

H. As a result of the introduction of the new tax, Nilufar fixed costs increased by \$140. Variable costs have also risen. What now are the variable costs per kilogram of dried apricots equal, if the total cost of 750 kilograms is \$3 165?

4.9. Problem

Zuhra during the summer holidays decided to work as an ice-cream seller. The company offers her two payment options. According to the first option, she will receive 60 kuruş (in Turkey, 1/100 lira) for each serving sold. According to the second option, Zuhra not receive anything over the first 30 servings and 96 kuruş for each successive sold serving. What decision should take Zuhra?

Solution

Zuhra's income (in kuruş), if she accepts the first option, will be expressed by the function $Y = 60q$, where q is the number of portions of ice-cream sold by Zuhra. In the second case: $Y = 96(q - 30)$.

The solution becomes clear if you draw graphs of functions $Y = 60q$ and $Y = 96q - 2\,880$.

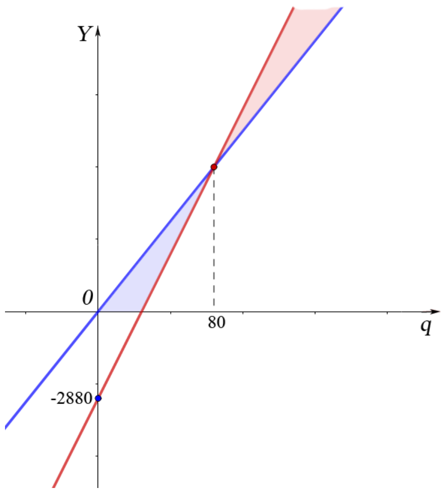


Figure 4.8

So if Zuhra says that she will not be able to sell more than 80 ice cream servings per day, then it is better to settle for the first option payment, otherwise – preferred the second option.

It is clear that the number 80 is the root of the equation $60q = Y = 96q - 2\ 880$.

Exercises 4.9

C. Aisha decided to work as a hamburger seller during the holidays. The company offers her two payment options. According to the first option, she will receive 4.5 soms for each portion sold. According to the second option, Aisha will not receive anything for the first 6 servings and 7 soms for each subsequent portion sold. What decision should Aisha make?

H. Salim decided to work during the holidays by renting bicycles. The company offers him two payment options. According to the first option, he will receive 22 soms for every hour of a rented bicycle. According to the second option, Salim will not receive anything in the first 9 hours and 33 soms for each subsequent hour of rent. What decision should Salim make?

4.10. Problem

Two publishers offer Thunger to publish his new book. The first promises to pay $8 for each book of the first 2,000 books sold, and $15 for each subsequent book sold. The second publisher will not pay anything for the first 500 books sold and $12 for each subsequent book sold. What decision should take Thunger?

Solution

Thunger's revenue (in dollars), if he accepts the first option, will be expressed by the function

$$Y = \begin{cases} 8q, & 0 \le q \le 2\,000; \\ 15q + b, & q > 2\,000, \end{cases}$$ where q is number of books sold.

Thunger's income when the number of books sold exceeds *2 000* is a function $Y = 15q + b$. The value of the coefficient b can be determined from the condition of the "joint" of the Thunger's income sub-functions: $8 \cdot 2\,000 = 15 \cdot 2\,000 + b$. Then, $b = -14\,000$.

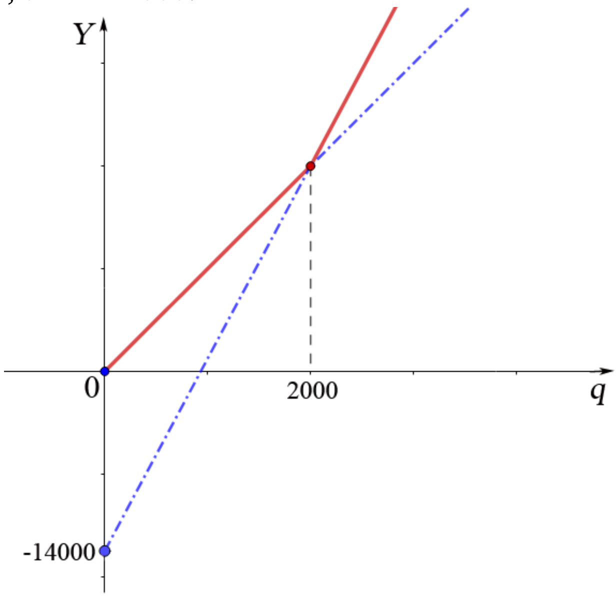


Figure 4.9

In the second case: $Y = 12(q - 500)$. Now let's take figure 9 and draw on it a graph of the function $Y = 12q - 6\,000$.

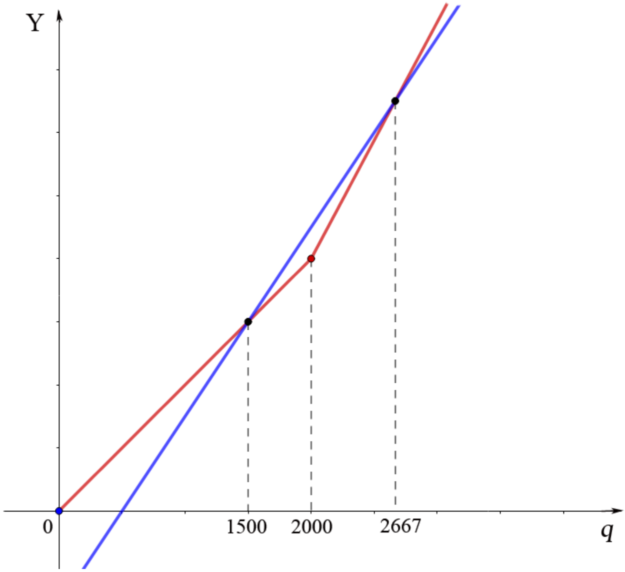


Figure 4.10

We get two points of intersection. The first is determined by the intersection of the graphs of the functions $Y = 8q$ and $Y = 12q - 6\,000$. It is clear that in this case $q = 1\,500$.

The second one is the graphs of the functions $Y = 15q - 14\,000$ and $Y = 12q - 6\,000$. In this case, $q \approx 2\,667$.

Thus, if Thunger believes that there will be a small or very high demand for a book (with $0 \le q \le 1\,500$ or $2\,667 \le q$), he should choose the first publisher, otherwise the second option is preferable.

Exercises 4.10

C. Aileen decided to work as a seller of hot dogs during the holidays. The company offers her two payment options. According to the first option, she will receive 4 soms for each of the first 35 sold hot dogs and 5 soms for each subsequent one. According to the second option, Aileen will not receive

anything for the first 5 hot dogs and 4.8 soms for each subsequent hot dog sold. What decision should Aileen make?
H. Two publishers are offering Said to publish his new book. The first promises to pay \$15 for each book from the first 1600 books sold, and \$24 for each subsequent book sold. The second publisher will not pay anything for the first 300 books sold and \$19 for each subsequent book sold. What decision should Said make?

4.11. Problem

The company buys T-shirts, puts on them images of Antalya and sells at a price of 4.9 lire (Turkey). Fixed costs (business licenses, advertising, etc.) is 5 700 lire. The firm can buy the first 2 250 T-shirts at a price of 3.5 lire, and it may have a discount of 2.1 lire for each subsequent T-shirt bought. Determine the break-even point.

Solution

Revenue is determined by the formula $R = 4.9q$, where q is the number of T-shirts sold. The total cost function will be $Y = \begin{cases} 3.5q + 5\,700, & 0 \le q \le 2\,250; \\ 2.1q + b, & q > 2\,250, \end{cases}$ where q is the number of T-shirts bought . Costs, when buying a large number of T-shirts is a function $Y = 2.1q + b$, where the value of the coefficient b can be determined from the condition of the "joint" of the total cost sub-functions:
$3.5 \cdot 2\,250 + 5\,700 = 2.1 \cdot 2\,250 + b$. Then, $b = 8\,850$.

The value of the break-even point depends on the total costs sub-function with which the revenue function intersect. If the break-even point on the first sub-function, then it is a solution to the equation $4.9q = 3.5q + 5\,700$. The root $q \approx 4\,071.4$ doesn't belong to the set $q \le 2\,250$ and can't be a break-even point. Consequently, the break-even point is determined by the intersection of the revenue function and the total cost function — by solving $4.9q = 2.1q + 8\ 850$. The root $q = 3\ 160$ indicates that the firm should engage in this type of business only if it is confident that it will be able to manufacture and sell more than *3 160* T-shirts with Antalya images.

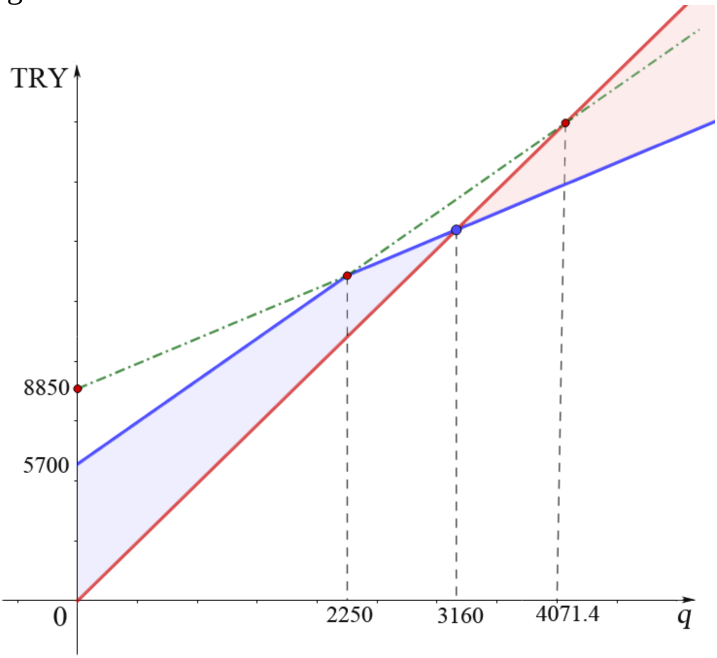


Figure 4.11

Exercises 4.11

C. The costs of preparation for the issuance and sale of a new type of product is \$1 480. Draw the graph and determine the profit zone, knowing that the average variable cost for the first 850 units is \$5.04, for the subsequent \$5.6, while a price equals \$6.1.

H. The costs of preparation for the issuance and sale of a new type of product is \$1 564. Draw the graph and determine the profit zone, knowing that the average variable cost for the first 440 units is \$5.5, for the subsequent \$5.2, while a price equals \$8.4.

4.12. From the previous considerations, a false conclusion may emerge: the more we produce and sell, the better. Any company can increase sales without changing the price only if it occupies a small market share. In a normal situation, you can increase sales by reducing the price.

Problem

Aladdin manufactures and sells lamps at a price of $p = 4 – 0.05q$ *dinars, where q is the number of lamps sold. The fixed costs of Aladdin (rent a workshop, taxes to the sultan, etc.) are 20 dinars, the cost of producing one lamp is 1.1 dinars. How many lamps should make and sell Aladdin in order to make a profit?*

Solution

As well as in the linear case, there are two methods for solving this problem.

The first method: Aladdin will fully recoup all his costs if his revenue is equal to the total costs. That is, there should be equality ***R = TC***: $(4 – 0.05q)q = 1.1q + 20$. It is very useful to draw graphs of revenue and total cost functions that define the left and right sides of the equation $4q – 0.05q^2 = 1.1q + 20$.

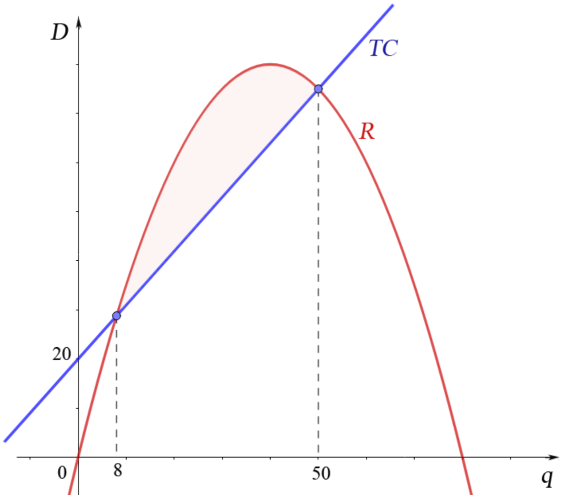


Figure 4.12

So, $q_1 = 8$; $q_2 = 50$. Thus it turned out that Aladdin will make a profit if he manufactures and sells from 8 to 50 lamps.

The second method: You need to write the profit function as the difference between the functions of revenue and total costs: $Pf = R – TC = (4 – 0.05q)q – (1.1q + 20)=$ $= – 0.05q^2 + 2.9q – 20$, and then equate it to zero. The roots of the resulting equation determine the profit zone:[8, 50].

Draw a graph of the profit function, which is an inverted parabola.

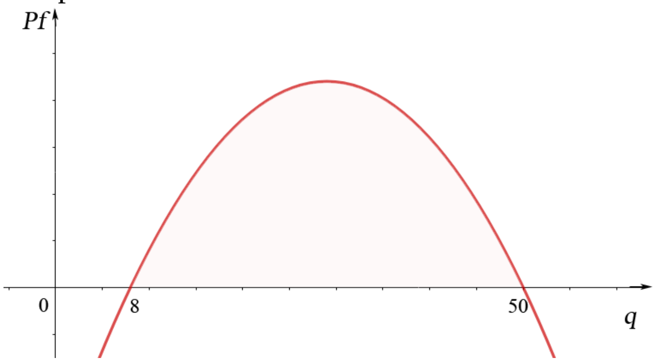


Figure 4.13

This mathematical model shows that you need to manufacture and sell more than a certain number of product units in order to cover fixed costs. At the same time, in order to sell a large number of goods, it is necessary to lower the price, which from some point will cease to cover variable costs.

Exercises 4.12

C. The costs of preparation for the issuance and sale of a new type of product is $5 780. Draw the graph and determine the profit zone, knowing that the average variable cost is $6.15, while a price (in $) is determined by the function $p = 28.25 - 0.02q$.

H. The costs of preparation for the issuance and sale of a new type of product is $4 620. Draw the graph and determine the profit zone, knowing that the average variable cost is $5, while a price (in $) is determined by the function $p = 24.4 - 0.02q$.

4.13. Problem

The costs of preparation for the release of a new model of trousers 24 000 soms. Determine the profit zone, knowing that the average variable cost of production pairs of trousers 500 soms, and each of them sell sells for 600 soms.

Solution

In order to solve the problem form the function of revenue

$$R = 600q \ (q \text{ is quantity}) \qquad (4.1)$$

and the cost function

$$C = 24\,000 + 500q. \qquad (4.2)$$

The functions R and C are linear.

The general form of a linear function is

$$y = kx + b. \qquad (4.3)$$

The graph of a linear function on a plane is a straight line. The coefficient k is the tangent of the angle between the OX-axis and the straight line (4.3).

It is usually referred to as ***slope***. When $k > 0$, a linear function is growing, when $k < 0$, it is decreasing, with $k = 0$, (4.3) is a horizontal line. The slope of the function (4.1) is the price of the pair of trousers, and the functions (4.2) – average variable cost (*AVC*). If the average variable costs are constant, they coincide with the marginal cost (*MC*).

The coefficient *b* — ***free term (y-intercept)*** is the coordinate point of intersection (4.3) with the *OY*- axis. In particular, the function *R* passes through the origin and function *C* through the point (in the problem 4.13 (*0, 24 000*)) determined by the value of the fixed costs (*FC*).

The profit function

$$Pf = R - C \qquad (4.4)$$

At a point of profit it takes a non-negative value. The set of points of profit form the profit zone. Negative values of the profit function are forming a losses zone.

Points dividing the profit zone and the losses zone are called ***break-even points.***

Returning to the problem. Knowing that at the break-even points the revenue coincides with the costs, we find this point: $600q = 24000 + 500q \Rightarrow q = 240$.

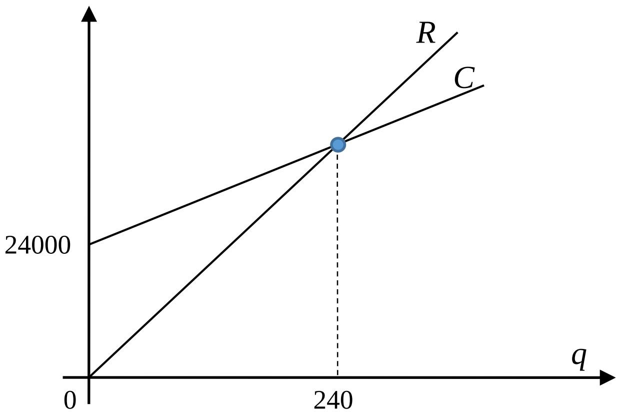


Figure 4.14

The corresponding figure demonstrates that the firm should produce and sell more than 240 units to make a profit.

Exersice 4.13

C. The costs of preparation for the issuance and sale of a new type of product is $780. Draw the graph and determine the profit zone, knowing that the average variable cost is $6.15, while the price is $7.65.

H. The costs of preparation for the issuance and sale of a new type of product is $4 650. Draw the graph and determine the profit zone, knowing that the average variable cost is $45, while the price is $57.

4.14. Problem LS

How will change the solution of problem 4.13, if it is necessary to take into account an additional lump-sum tax 5 000 soms.

Lump - sum tax will be charged as a fixed sum of money. It is usually regarded as a fee for the patent.

A firm perceives the lump tax as additional fixed costs. As a result, changing the cost function: $C_L = 24000+500q+5000.$

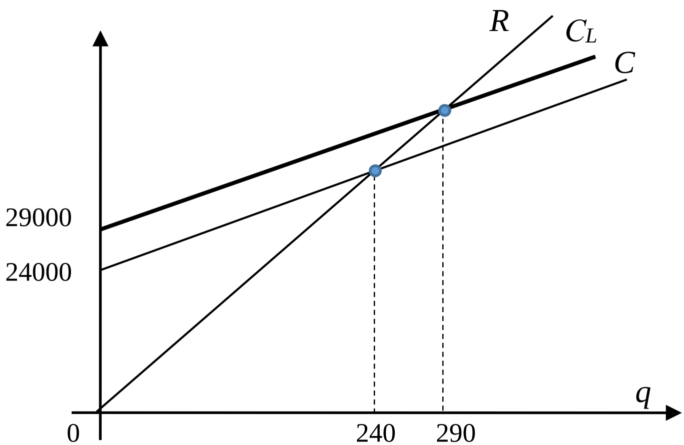


Figure 4.15

(As mentioned above, the change in the free term of the linear equation leads to a parallel shift of the line.)
Accordingly, the break-even point satisfies the equation $600q = 29\ 000 + 500q$ and equals to 290. So, the lump-sum tax, as would be expected, required to increase production and sales to achieve the profit zone.

Exersice 4.14

C. How will change the solution of problem 4.13.C, if it is necessary to take into account an additional lump-sum tax $45.

H. How will change the solution of problem 4.13.H, if it is necessary to take into account an additional lump-sum tax $150.

4.15. Problem E

How will change the solution of problem 4.13, if it is necessary to consider an additional excise tax 20 soms per unit.

Excise tax is a fixed amount of money that is paid for each unit sold goods or services. The introduction or modification of excise duty leads to a corresponding change in the marginal cost *MC*.

Solution

If the marginal cost of *MC* are constant, they coincide with the average variable costs. Therefore $AVC_E = 500 + 20$. Then, $C_E = 24\ 000 + 520q$.
As a consequence, we find the break–even point from the equation $600q = 24\ 000 + 520q \Rightarrow q = 300$.

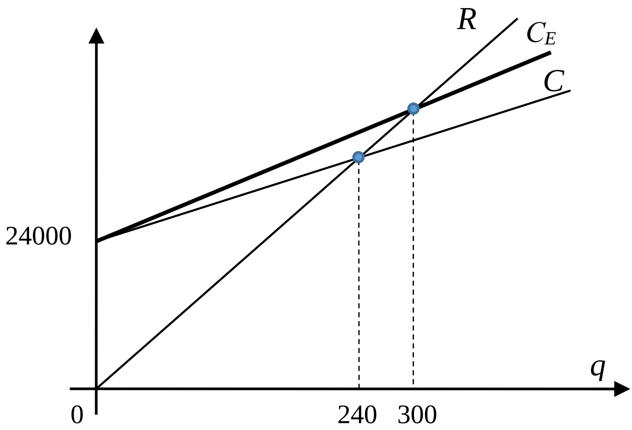


Figure 4.16

(Changing the cost function has changed the slope of the corresponding line at this case.)

Exersice 4.15

C. How will change the solution of problem 4.13.C, if it is necessary to take into account an additional excise tax \$0.3 per unit.

H. How will change the solution of problem 4.13.H, if it is necessary to take into account an additional excise tax \$2.7 per unit.

4.16. In solving the problems of the above style, we can come to a false idea: "The more is done, the better." Therefore, along with such tasks, it is necessary to consider the problem in which the price depends on the quantity.

Problem

The costs of preparation for the release of a new model TV is \$210 000. Determine the profit zone, knowing that the average variable cost of production of TV is \$400, and the price is determined by the function $p = 1\,400 - q$.

Solution

To solve this problem we proceed in the same way as in the solution of the problem 4.13: form the function of revenue

$$R = (1\,400 - q)\,q, \qquad (4.5)$$

and the cost function

$$C = 210\,000 + 400q. \qquad (4.6)$$

The function (4.5) is a special case of a quadratic function, which is written in a general way as

$$y = mx^2 + kx + b. \qquad (4.7)$$

The graph of a quadratic function is a parabola on the plane, directed branches up for $m > 0$: ∪ and down for $m < 0$: ∩.

When plotting a parabola it is useful to remember that the coefficient b — *free term* is the point of intersection of the parabola with the axis *OY*; the roots x_1 and x_2 of the equation

$$mx^2 + kx + b = 0 \qquad (4.8)$$

determine the coordinates of the points of intersection of the parabola (4.7) with the *OX* axis.

Values x_1 and x_2 can be found from the formulas

$$x_1 = \frac{-k - \sqrt{D}}{2m} \quad \text{and} \quad x_1 = \frac{-k + \sqrt{D}}{2m},$$

where $D = k^2 - 4mb$.

If $D < 0$ and $m > 0$ the parabola is above the horizontal axis $\underline{\cup}$, and if $D < 0$ and $m < 0$ — below it: $\overline{\cap}$.

Equating the function (4.5) and (4.6): $(1\,400 - q)q = 210\,000 + 400q$ we come to the quadratic equation $-q^2 + 1\,000q - 210\,000 = 0$, which has roots $q = 300$ and $q = 700$.

From the graph

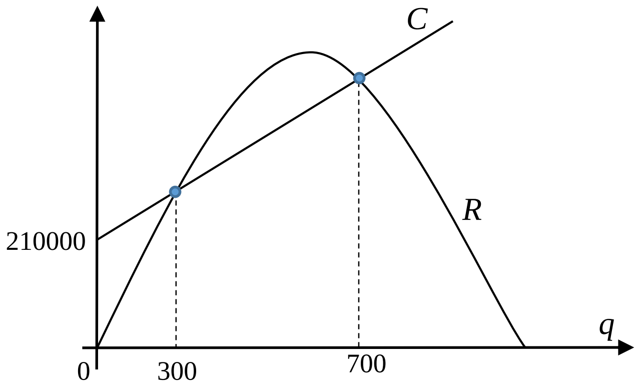


Figure 4.17

it is clear that in order to make a profit the firm must produce and sell more than 300 units sold, but at the same time, the volume of production must not exceed 700 units.

Exersice 4.16

C. The costs of preparation for the issuance and sale of a new type of product are \$5 400. Draw the graph and determine the profit zone, knowing that the average variable cost is \$9, while the price (in \$) is $p = 30 – 0.02q$.

H. The costs of preparation for the issuance and sale of a new type of product are \$11 550. Draw the graph and determine the profit zone, knowing that the average variable cost is \$12.5, while the price (in \$) is $p = 61 – 0.05q$.

4.17. Problem Pf

How will change the solution of problem 4.16, if the profit tax 20% must also be taken into account.

Solution

Since the value of after-tax profit is $Pf_1 = (1 – 0.2)Pf$, and the equation that determines the boundary of the profit zone $(1 – 0.2)\, Pf = 0$ is equivalent to the equation $Pf = 0$, the solution of the problem 4.17

coincides with the solution of the problem 4.16. Changing only the value of after-tax profits.

Therefore, the profit tax, in theory, does not affect the volume of the market. The difference between theory and practice is conditioned by the fact that the profit tax will be charged with the economic and accounting profit.

Exersice 4.17

C. How will change the solution of exercise 4.16.C, if the profit tax 30% must also be taken into account?

H. How will change the solution of exercise 4.16.H, if the profit tax 10% must also be taken into account?

4.18. Problem

The costs of preparation for the issuance and sale of a new type of product are $5 000. Determine the profit zone, knowing that the average variable cost for the first 128 units are $10 for subsequent $12.5, while the price is $60.

Solution

The revenue function is $R = 60q$ and the function of the total costs for the first 128 units is $C = 5\,000 + 10q$. From equation $60q = 5\,000 + 10q$ find that the solution of the problem is the interval $(100, +\infty)$.

Exersice 4.18

C. The costs of preparation for the issuance and sale of a new type of product are $330. Draw the graph and determine the profit zone, knowing that the average variable cost for the first 320 units is $5, for subsequent $4.6, while the price (in $) is *6.2*.

H. The costs of preparation for the issuance and sale of a new type of product are $920. Draw the graph and determine the profit zone, knowing that the average variable cost for the first 500 units is $4.4, for subsequent $4.7, while the price (in $) is *6.7*.

4.19. Problem LS

How will change the solution of problem 4.18, if it is necessary to take into account an additional lump-sum tax value of \$2 920.

Solution

The introduction of tax changes the cost function:

$C_L = 5\,000 + 10q + 2\,920.$

Then, a solution of equation $60q = 7\,920 + 10q$ is $q = 158.4$. But this number does not determine the profit zone, as a function of C_L determines the costs only for the first 128 units sold. In order to get the right answer we should construct a cost function for other values of q. Since the total cost of production of 128 units sold are

$7\,920 + 10\cdot 128 = 9\,200$, and the average variable cost for subsequent units equal to 12.5, the corresponding function can be determined by substituting the coordinates of the point $(128, 9\,200)$ in equality

$C_{L1} = 12.5q + b$: $9200 = 12.5\cdot 128 + b \Rightarrow \quad b = 7600.$

So, the total costs of production and sales after the first 128 units will be determined by function $C_{L1} = 12.5q + 7\,600.$

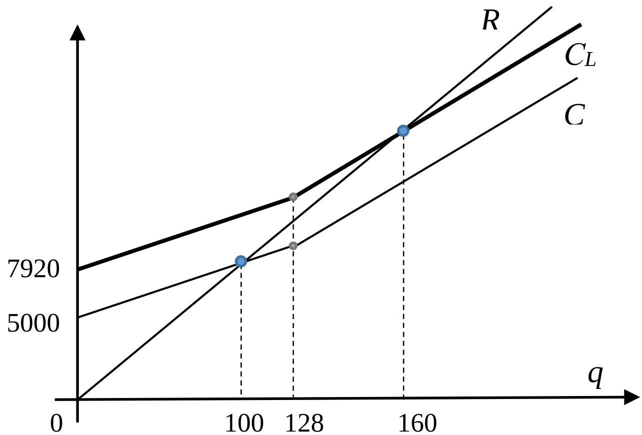


Figure 4.18

Hence, equating the functions of revenue and expenses,

$60q = 12.5q + 7\ 600$, and solving the resulting equation, we find that for profit it is needed to produce and sell more than 160 units.

Exersice 4.19

C. How will change the solution of exercise 4.18.C, if it is necessary to take into account an additional lump-sum tax value of $210.

H. How will change the solution of exercise 4.18.C, if it is necessary to take into account an additional lump-sum tax value of $340.

4.20. Problem Q

How will change the solution of problem 4.18, if the price is determined by the function $p = 80 - 0.2q$.

Solution

Equating the function of revenue and cost function C, we obtain a quadratic equation

$(80 - 0.2q)q = 5\ 000 + 10q \Rightarrow -0.2q^2 + 70q - 5\ 000 = 0$, having roots $q_1 = 100$ and $q_2 = 250$. Since $q_2 > 128$, it does not belong to the domain of the function C. Therefore, in order to find the second boundary of the zone of profit, you need to write a function that determines the costs for $q > 128$. Since $C(128) = 5\ 000 + 10 \cdot 128 = 6\ 280$, and the slope, determined by the average variable cost is equal to 12.5, the line $C_1 = 12.5q + b$ passes through the point $(128, 6\ 280)$. Then $6\ 280 = 12.5 \cdot 128 + b \Rightarrow b = 4\ 680$. Now equating the function of revenue and cost function $(80 - 0.2q)\ q = 4\ 680 + 12.5q$, and solving the resulting quadratic equation $-0.2q^2 + 67.5q - 4\ 680 = 0 \Rightarrow$ $\Rightarrow q_1 = 97.5$ and $q_2 = 240$, we find the upper boundary of the profit zone $q = 240$.

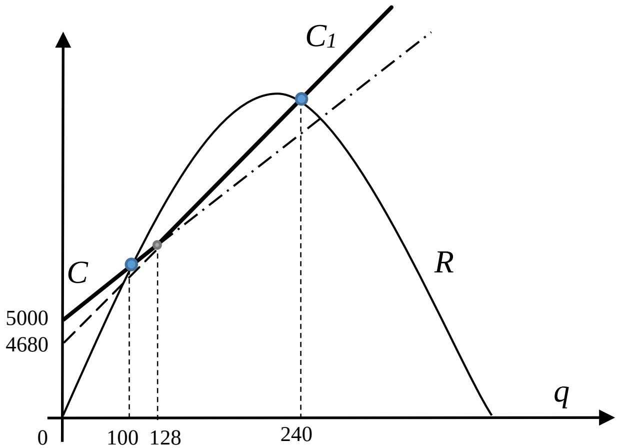


Figure 4.19

Exersice 4.20

C. The costs of preparation for the issuance and sale of a new type of product are \$150 000. Draw the graph and determine the profit zone, knowing that the average variable cost for the first 3 575 units is \$1 000, for subsequent \$600, while the price (in \$) is $p = 9\,000 - q$.

H. The costs of preparation for the issuance and sale of a new type of product are \$4 000. Draw the graph and determine the profit zone, knowing that the average variable cost for the first 60 units is \$30, for subsequent \$23, while the price (in \$) is $p = 160 - q$.

Summary

1. Determine the profit zone, knowing the cost function $C = 200 + 5q$ and price:

a) $p = 21$; b) $p = 35 - q$.

2. Determine the profit zone, knowing the cost function $C = 70 + 9q$ and price:

a) $p = 16$; b) $p = 28 - q$.

3. The costs of preparation for the release of a new type of product are 60 thousand soms. Determine the profit zone, knowing that the average variable cost of production per unit of goods 2 thousand soms, and the price (in thousand soms) is determined by the function:

a) $p = 8$; b) $p = 10 - 0.2q$.

4. The costs of preparation for the release of a new type of product are180 thousand soms. Determine the profit zone, knowing that the average variable cost of production per unit of goods 2.4 thousand soms, and the price (in thousand soms) is determined by the function:

a) $p = 6.9$; b) $p = 21 - 0.3q$.

5. Determine the profit zone, knowing the cost function $C = 540 + 20q + 0.1q^2$ and price

a) $p = 37.5$; b) $p = 50 - 0.3q$.

6. Determine the profit zone, knowing the cost function $C = 650 + 31q - 0.1q^2$ and price

a) $p = 39$; b) $p = 54 - 0.3q$.

7. The machine cost $2 400 written off after 7 years with a residual value is $195. Determine the book value of the machine after 5 years and 8 months using the straight line depreciation.

8. Price (in KGS) at November 21, 2003 of one euro was equal to 52.1039, the one US dollar 43.9222. Price at March 12, 2004 was 52.7734 and 43.2622, consequently. How much will cost one euro when the price of the one US dollar is equal to 43.5151? Assume the linear relation takes place.

9. A student who spent 9.5 hours to prepare for tests, got 27 points, having spent 13 hours 21 minutes — 33 points. How much time must be spent in order to get 45 points if a linear relation takes place?

10. When the sales volume reach 1 750 000 soms manager receives a premium of 21 000 soms, at 1 920 000 soms

receives 244 000 soms. What will be the premium when the sales reach 1 956 000 soms, if the linear relation takes place?
11. The costs of preparation for the release of a new type of goods are $2 100. Determine the profit zone, knowing that the average variable cost of production is $50 and the price is determined by the function $p = 150 – q$.
11a. How will the answer change if in content of the problem 11 there is an additional lump-sum tax $300?
11b. How will the answer change if in content of the problem 11 there is an additional excise tax $7.5?
12. Brian plans to begin production of a new type of product. According to the assumptions the fixed costs are $12 000, the average variable cost of production for the first 400 units sold is $40, for subsequent units is $36, the price is $88.
a) Help Brian to identify the profit zone.
b) After a few days it became clear that it is necessary to pay the lump-sum tax $7 460. How will the answer change in this case?
13. Solve the problem 12a, with the function of the price $p = 105.2 – 0.084q$.
14. Determine the profit zone, knowing that the fixed costs are $2 520, the average variable cost of production is $65 and the price is $84.
15. Determine the profit zone, knowing that the fixed costs are $9 000, the average variable cost of production is $260 and the price is $p = 400 – 0.5q$.
15a. How will the answer change if in content the problem 15 is additionally introduced a lump-sum tax of $600.
15b. How will the answer change if in content the problem 15 is additionally introduced an excise tax of $5?
16. Makhabat plans to begin production of a new type of product. She assumptions fixed costs are $30 060, the average variable cost of production for the first 900 units sold is $60, for subsequent units is $64, the price is $96.

a) Help to identify Makhabat's profit zone.

b) After a few days it became clear that it is necessary to pay lump-sum tax of $7 460. How will the answer change in this case?

17. Solve the problem 16a, with the function of the price $p = 126.035 – 0.035575q$.

18. Determine the profit zone, knowing that the fixed costs are $800, the average variable cost of production unit price is $40 and the price is $p = 72 – 0.24q$.

19. A writer can give the right to publish his new book, to one of the two firms. The firm P is offering $6 for each of the first 30 000 books sold and $8.5 for each additional book sold. The firm Q will not pay anything for each of the first
a) 4 000; b) 20 000 books sold, and $9 for each additional book sold. What decision should be taken?

20. Meerim can sell ice cream in one of the two firms. She will receive $0.1 for each of the first 200 sold portions and $0.18 for each additional portion sold in firm M. She will not receive anything for each of the first 50 servings sold and $0.15 for each additional portion sold in the firm K. What decision should be taken?

21. The costs of preparation for the issuance and sale of a new type of product are $4 250. Draw the graph and determine the profit zone, knowing that the average variable cost is $6.7, while the price (in $) is determined by the function is $p = 26 – 0.02q$.

22. The costs of preparation for the issuance and sale of a new type of product is $1 080. Draw the graph and determine the profit zone, knowing that the average variable cost for the first 400 units is $5, for the subsequent is $4.6, while the price is $6.2.

23. The costs of preparation for the issuance and sale of a new type of product are $27 000. Draw the graph and determine the profit zone, knowing that the average variable cost is $45, while the price (in \$) is determined by the function is $p = 150 - 0.1q$.

24. The costs of preparation for the issuance and sale of a new type of product are \$4 600. Draw the graph and determine the profit zone, knowing that the average variable cost for the first 555 units is \$22, for the subsequent is \$24, while the price is \$33.5.

25. The costs of preparation for the issuance and sale of a new type of product are \$2 240. Draw the graph and determine the profit zone, knowing that the average variable cost for the first 90 units is \$7, for subsequent is \$7.8, while the price (in \$) is $p = 75 - 0.5q$.

§5. General Equation of a Straight Line

5.1. Equation of the line in a parametric form

The Cartesian coordinates allow us to describe geometrical objects with the help of equations, inequalities, or their systems.

An equation (inequality, system) is the equation of a geometrical object, where provided coordinates of all the points of the object satisfy the equation. For example, equation

$(x - 2)^2 + (y + 1)^2 = 0$, is the equation of the point $(2, -1)$, because the sum of term's squares equals zero, if and only if each term equals zero. Equation $(x - 2)^2 + (y + 1)^2 = 2^2$ describes a set of points, which are located at a distance of 2 (see the formula of segment, vector length) from the point $(2, -1)$. The set of points is called a circle with the center in the point $(2, -1)$ and the radius of 2.

One of the basic postulates of Euclidean geometrics, is the assertion that a straight line is determined by two points.

(Select two random points, and then draw a line through those points). It follows that in order to write an equation of a straight line, it is sufficient to have coordinates of two points. So, let $K(x_1, y_1)$ and $L(x_2, y_2)$ be the points of the straight line *m*. Then the vector

$\overline{KL} = (x_2 - x_1; y_2 - y_1)$ – defines the direction of the line *m*, and is called directing vector of the line *m*. If (x, y) are the coordinates of a random point *X*, located on *m*, then the $\overline{KX}$ can be obtained by multiplying $\overline{KL}$ by a number *p*:

$$(x - x_1; y - y_1) = p(x_2 - x_1; y_2 - y_1). \qquad (5.1)$$

We can rewrite the equation (5.1) as:

$$\begin{cases} x = x_1 + p\left(x_2 - x_1\right), \\ y = y_1 + p\left(y_2 - y_1\right); \end{cases} \qquad (5.2)$$

and get the equation of the line *m* in a parametric form.

Problem

Let the straight line l be defined by the points A(3,–2) and B(5, 6). And we should determine the coordinates of the 5 points, which are located on l.

Solution

We write the equation *l* in a parametric form:

$$\begin{cases} x = 3 + p \cdot 2, \\ y = -2 + p \cdot 8. \end{cases} \qquad (5.3)$$

Having written the equation (5.3), we almost finished the task, because each value of the parameter *p* determines the coordinates of the points. So, if $p = -1$, we get a point, located on *l* with the coordinates $(1, -10)$; if $p = 2$, we get a point $(1, -10)$ and so on.

We note that if $p = 0$, from to the system (5.3) we get the coordinates of the point *A*, and if $p = 1$, we get the coordinates of the point *B*.

Exercises 5.1

C. The straight line m be defined by the points $C(2, -1)$ and $D(-3, 4)$. Determine the coordinates of the 5 points, which are located on m.

H. The straight line n be defined by the points $E(-3, -2)$ and $F(5, 1)$. Determine the coordinates of the 5 points, which are located on n.

5.2. Equation of a straight line through two given points

Expressing the parameter p from the system (5.2), and equating obtained expressions, we get a different form of the equation of the line m:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}. \qquad (5.4)$$

This equation is often called equation of a straight line through two given points.

Note We note that equation (5.4) is an equality ratio. In this situation, it is possible to have a zero in the denominator. For example, equation of a line, written in the form $\frac{x-3}{3}=\frac{y-2}{0}$

is equal to the equation

$3(y-2) = 0(x-3) \Leftrightarrow y-2 = 0.$

Straight line l from the problem 1, written in the form (5.4):

$\frac{x-3}{2}=\frac{y+2}{8}.$

Exercises 5.2

C. Write an equation of the straight line k which defined by the points $(5, -11)$ and $(3, 0)$.

H. Write an equation of the straight line s which defined by the points $(1, 7)$ and $(-2, 9)$.

5.3. Parallel lines

Equation (5.4) doesn't change when we multiply denominators by the same number. It is equivalent to the situation when instead of vector $\overline{KL}$ we take another vector, which is parallel to $\overline{KL}$. Therefore, we get:

$$\frac{x-x_1}{m_1}=\frac{y-y_1}{m_2} \qquad (5.5)$$

Here (m_1; m_2) are coordinates of the vector which is parallel to a straight line m. The vector is also named a directing vector.

Vector (2; 8) is directing for all the lines which are parallel to a line $\frac{x-3}{2}=\frac{y+2}{8}$. Therefore, a line n, which is parallel to the line $\frac{x-3}{2}=\frac{y+2}{8}$ and passes through the point $C(1, 7)$, can be written in the following form: $\frac{x-1}{2}=\frac{y-7}{8}$.

Exercises 5.3

C. Write an equation of the straight line, which passes through the point (11, – 2) and parallel to the line $\frac{x-3}{7}=\frac{y+12}{-33}$.

H. Write an equation of the straight line, which passes through the point (– 3, 15) and parallel to the line $\frac{x+6}{11}=\frac{y-2}{3}$.

5.4. Angular coefficient of a straight line

Let's rewrite the equation (5.4):

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad (5.6)$$

A number $\dfrac{y_2 - y_1}{x_2 - x_1}$ is a tangent of the angle between *OX* axis and a straight line (5.6).

It is called an angular coefficient of a straight line or ***slope***.

Let's designate it as *k*, and write (5.6) in the form:

$$y - y_1 = k(x - x_1). \qquad (5.7)$$

Equation (5.7) tells us that the straight line is definitely determined by the slope and the point lying on the line. It follows, that the parallel lines have the same slope.

Problem

1) *Write an equation of the straight line, which passes through the point (1, – 21) and parallel to the line* $y + 7 = -5(x - 2)$.

2) *Write an equation of a straight line, which forms* 15^0 *angle with the line s:* $y = x - 7$, *and passes through the point (2.72, 3.14).*

Solution

1) The straight line $y + 7 = -5(x - 2)$ has the slope (-5) and passes through the point $(2, -7)$. Because the parallel lines have the same slope, answer: $y + 21 = -5(x - 1)$.

2) Since the slope of the line *s* equals 1 and is a tangent of the 45^0 angle, desired equation has a slope $tg(45^0 + 15^0) = tg60^0 = \sqrt{3}$.

Therefore, it can be written in the following form $y - 3.14 = \sqrt{3}\,(x - 2.72)$.

Important note In an analytic geometry, the formula $tg\alpha = \dfrac{k_2 - k_1}{1 + k_1 k_2}$ is often used, which allows to determine the angle between straight lines $y = k_1x + b_1$ and $y = k_2x + b_2$.

In our opinion, the formula can be painlessly excluded from the course, because k_1 and k_2 allow us to find the angle between straight lines through the arctangent.

Exercises 5.4

C. Write an equation of the straight line, which passes through the point *(– 9, 27)* and parallel to the line $y - 17 = 12(x - 0.5)$.

H. Write an equation of the straight line, which passes through the point *(1.1, 2.1)* and parallel to the line $y + 37 = 1.5(x + 32)$.

5.5. Linear function

Let's write the equation of the straight line (5.7) in the form:

$$y = kx + b. \qquad (5.8)$$

The number b is a coordinate of the intersection point of the line m with the OY axis.

If there's a relation in the form (5.8) between x and y, we say that there's a linear dependence. The function (5.8) is called linear. It follows that the graph of the linear function is a straight line.

Note Equation (5.8) describes all straight lines on the plane, except vertical ones. In this case equation of the straight line is $x = a$.

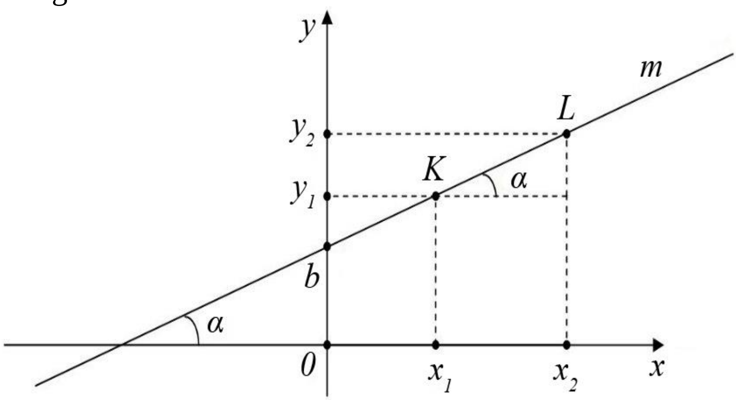


Figure 5.1

Problem

1) *Air temperature is* 12^0 *at 2 PM,* 7^0 *at 6 PM. Determine the air temperature at 2:30 PM, considering linear dependence.*

2) *Suppose "Pear" company produced 240 computers in 2008, 1 200 computers in 2014; we also know that the year to year production was growing in equal amounts.*

How many computers were produced in 2010, 2013?

Solution

1) Since we have a linear dependence, air temperature T and time relate in the following way: $T = kt + b$. We substitute given information: $\begin{cases} 12 = k \cdot 2 + b, \\ 7 = k \cdot 6 + b, \end{cases}$ subtract the first equation from the second one, and get $5 = -4k$. Then, $k = -1.25$, consequently $b = 14.5$.

So, in the context of our problem (within the limits of assumptions), temperature and time are related the following way: $T = -1.25t + 14.5$.

Therefore, at 2:30 PM air temperature equals $T(2.5) = -1.25 \cdot 2.5 + 14.5 = 11.375^0$.

2) Production growth of equal amounts means that the volume of production (production output) is the linear function, which depends on the year of production: $Y = kn + b$.

We substitute given information: $\begin{cases} 240 = k \cdot 2008 + b, \\ 1200 = k \cdot 2014 + b, \end{cases}$

subtract the first equaiton from the second one, and get $960 = -6k$. Then, $k = 160$, consequently we get $b = -321\,040$.

So, we obtained equation which connects the year with the production volume $Y = 160n - 321\,040$.

Now, in order to answer the questions, we simply substitute required number into n:

in 2010: $Y = 160 \cdot 2010 - 321\,040 = 560;$

in 2013: $Y = 160 \cdot 2013 - 321\ 040 = 1\ 040.$

Obtained equation can also be used to determine future production volumes, provided that the rate of growth is the same. For example, in 2019 excepted production

$Y = 160 \cdot 2019 - 321\ 040 = 2\ 000$ computers.

Note We shouldn't entirely rely on the number 2 000, because it is based upon assumption that the growth of amount of production is fixed, which is very unlikely.

Exercises 5.5

C. Profit of Kodir's firm on revenue \$75 000 is \$12 000, on revenue \$125 000 is \$32 000. Assuming a linear relationship, determine the amount of profit for revenue \$90 000.

H. After buying 15 T-shirts, Said had 6 900 soms. After he bought another 25 T-shirts, he had 3 400 soms. How much money did Said have at the beginning if the price of a T-shirt is 140 soms?

5.6. General form of straight line equation on a plane.

A straight line, due to its simplicity, is one of the most important tools in scientific research. Now, let's write the other versions of a straight line's equation.

Let $(A; B)$ be the coordinates of a vector, which is perpendicular to the straight line m, and (x_1, y_1) are the coordinates of the point K, which lies on the line m. Then, an arbitrary point with the coordinates (x, y) on the line m forms a vector $(x - x_1; y - y_1)$, with the point $K(x_1, y_1)$, which is perpendicular to $(A; B)$. Therefore, the dot (scalar) product is zero:

$$A(x - x_1) + B(y - y_1) = 0. \qquad (5.9)$$

Having rearranged the equation (5.9), we get

$$Ax + By = C. \qquad (5.10)$$

This equation is called a general form of straight line equation on a plane.

Problem

Write an equation of a triangle height, dropped to the side which lies on the straight line $-3x + 7y = 11$*. The others sides lie on the straight lines* $5x - 7y = 1$ *and* $3x + 4y = 23$*.*

Solution

The height drops from the vertex of the triangle, which is formed by the intersection of two straight lines $5x - 7y = 1$ and $3x + 4y = 23$. Therefore, the coordinates of the vertex can be found through the system of equations:

$$\begin{cases} 5x - 7y = 1, \\ 3x + 4y = 23, \end{cases} \Rightarrow \begin{cases} x = 5, \\ y = 2. \end{cases}$$

The coefficients of the third equation $(-3;\ 7)$ are the coordinates of the directing vector of the desired straight line. Therefore, according to (5.5), desired equation is:

$$\frac{x-5}{-3} = \frac{y-2}{7}.$$

Since the equation of a straight line is often written in the form of linear function, let's determine the slope of the perpendicular line: we write the equation of the line $y = kx + b$ in the following form $kx - y + b = 0$.

Then $(k;\ -1)$ are the coordinates of the vector, which is perpendicular to the straight line. If we write the desired equation $y = k_1x + b_1$ in the following form $k_1x - y + b_1 = 0$, then the scalar product of its coefficients $(k_1;\ -1)$ with the coefficients of the initial equation is 0: $k \cdot k_1 + (-1)(-1) = 0$. Hence, it follows that the slope of the line, which is perpendicular to the line $y = kx + b$ equals $(-1/k)$, which means that it's opposite in value and sign.

Thus, since the equation $-3x + 7y = 11$ can be rewritten in the form $y = (3/7)\,x + 11/7$, the slope of the height is $(-7/3)$. Since the height drops from the vertex whose coordinates are $(5,\ 2)$, the equation of height:

$y - 2 = (- 7/3)(x - 5)$. It is easy to see that this equation is equivalent to one obtained previously.

Exercises 5.6

C. Write an equation of a triangle height, dropped to the side which lies on the straight line $5x - 7y = 1$. The others sides lie on the straight lines $-3x + 7y = 11$ and $3x + 4y = 23$.

H. Write an equation of a triangle height, dropped to the side which lies on the straight line $3x + 4y = 23$. The others sides lie on the straight lines $-3x + 7y = 11$ and $5x - 7y = 1$.

5.7. Problem

We've spent $420 in order to buy labor and capital. How many units of labor and capital do we have, if 1 unit of labor costs $5, and 1 unit of capital costs $7?

Solution

In our case, 1 unit of labor is 1 work hour, and 1 unit of capital is the rent of equipment for 1 hour.

Hence, we have the following equation $5x + 7y = 420$. In economic theory, this line is called isocost — fixed cost line.

We divide the equation by 420 and get: $\frac{x}{84} + \frac{y}{60} = 1$. This form is very useful for drawing, plotting – described straight line connects the point 84 lying on OX axis and the point 60 lying on OY axis.

The following form of straight line equation

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad (5.11)$$

is called two-intercept form. It shows that the line passes through the points $(a, 0)$ and $(0, b)$.

Exercises 5.7

C. Notebook's price is 4 soms, the price of the pen is 10 soms. Write the isocosts equation assuming 200 soms have been spent. How will the isocost change if the price of a notebook rises to 5 soms? Draw both isocosts in the same coordinate system.

H. The price of the cupcake is 25 soms, the price of a cup of tea is 10 soms. Write the isocosts equation assuming 250 soms have been spent. How will the isocost change if we assume that 300 soms have been spent? Draw both isocosts in the same coordinate system.

5.8. Linear equations in several variables

We note that, many of the methods used to describe a straight line on a plane can also be used in multidimensional planes. Let $A(a_1, a_2, \ldots, a_n)$ and $B(b_1, b_2, \ldots, b_n)$ be the points on the straight line l.

Then $\overline{AB} = (b_1 - a_1; b_2 - a_2; \ldots; b_n - a_n)$ – is the vector lying on the straight line l. If $(x_1, x_2, \ldots, x_n)$, are the coordinates of an arbitrary point X, which lies on l, then the vector $\overline{AX}$ can be obtained by multiplying $\overline{AB}$ by the number p:

$$(x_1 - a_1; x_2 - a_2; \ldots; x_n - a_n) = p(b_1 - a_1; b_2 - a_2; \ldots; b_n - a_n). \quad (5.12)$$

Let's rewrite the equation (5.12) in the form

$$\begin{cases} x_1 = a_1 + p(b_1 - a_1); \\ x_2 = a_2 + p(b_2 - a_2); \\ \quad \ldots\ldots\ldots\ldots \\ x_n = a_n + p(b_n - a_n) \end{cases} \quad (5.13)$$

and we get an <u>equation of the straight line l in a parametric form.</u>

Problem

Let the straight line l be defined by the points $A(5, 3, -2)$ and $B(5, -6, 3)$. And we should determine the coordinates of the 5 points, which are located on l.

Solution

We write the equation l in a parametric form:

$$\begin{cases} x = 5 + p \cdot 0; \\ y = 3 + p \cdot (-9); \\ z = -2 + p \cdot 5. \end{cases}$$

Having written the equation in a parametric form, we actually solved the problem, because each value of the parameter *p* determines the coordinates of the points. So, if *p* = – *1*, we get a point, located on *l* with the coordinates (*5, 12, – 7*); if *p* = *2*, we get a point (*5, –15,– 12*) and so on. We note that if *p* = *0*, we get the coordinates of the point *A*, and if *p* = *1*, we get the coordinates of the point *B*.

Exercises 5.8

C. Determine the coordinates of any 4 points, which are located on the straight line, which passes through the points *(– 5, – 1, 21)* and *(3, 0, – 5)*.

H. Determine the coordinates of any 4 points, which are located on the straight line, which passes through the points *(12, 51, – 1)* and *(33, – 10, 43)*.

5.9. Equation of the line passes through the points

Having expressed *p* from the equation system (5.13), we get another form of the straight line equation *l*:

$$\frac{x_1 - a_1}{b_1 - a_1} = \frac{x_2 - a_2}{b_2 - a_2} \ldots = \frac{x_n - a_n}{b_n - a_n} \qquad (5.14)$$

Problem

*Write the equation of the line **k,** which passes through the points (– 7, 2, – 4, 8) and (– 4, 3, 4, 1).*

Solution

Coordinates of the directing vector:
(– 4 – (– 7); 3 – 2; 4 – (– 4); 1 – 8) = (3; 1; 8; – 7).
Then, according to the formula (5.14), we get the equation of the line ***k***:

$$\frac{x_1-(-7)}{3}=\frac{x_2-2}{1}=\frac{x_3-(-4)}{8}=\frac{x_4-8}{-7}\ .$$

Exercises 5.9

C. Write the equation of the straight line**,** which passes through the points *(7, 12, 4)* and *(4, – 3, – 1).*

H. Write the equation of the straight line**,** which passes through the points *(17, – 2, 3, 18)* and *(–1, – 7, 4, 11).*

5.10. Straight line parallel to given

In (5.14), we can multiply all denominators by the same number, in other words we take another parallel vector. And, we get the following equation

$$\frac{x_1-a_1}{m_1}=\frac{x_2-a_2}{m_2}\ ...=\frac{x_n-a_n}{m_n}. \qquad (5.15)$$

Here ($m_1; m_2; ...; m_n$) – coordinates of the vector, which is parallel to the line *l.* It is called a directing vector.

Problem

Write the equation of the line ***m,*** *which passes through the 2 points: (2, – 4, 8) and (4, 4, 1), and also equation of the line which is parallel to* **m**, *and passes through the point C(1, 7, – 2.5).*

Solution

Coordinates of the directing vector:
(4 – 2; 4 – (– 4); 1 – 8) = (2; 8; – 7). Then, according to the formula (5.15), we get the equation of the line ***m***:

$$\frac{x-2}{2}=\frac{y-(-4)}{8}=\frac{z-8}{-7},$$

and the parallel line is described by the following equation

$$\frac{x-1}{2}=\frac{y-7}{8}=\frac{z+2.5}{-7}.$$

Exercises 5.10

C. Write the equation of the straight line**,** which parallel line from exercise 5.9**.C** and passes through the point *C(21, – 7, 1.5).*

H. Write the equation of the straight line**,** which parallel line from exercise 5.9.**H** and passes through the point *C(12, 1.7, – 5, 6).*

Summary

1. The points *A(5, – 7), B(– 3, 1), C(2, 3)* are given. Find:

a) the equation of a straight line *AB;*

b) the equation of a straight line *CD* which is a parallel to *AB*;

c) the equation of a straight line *CH*, which is a perpendicular to *AB*;

d) the coordinates of 5 points of a straight line *AB*;

e) the slope of a straight line *CK*, where a point *K* divide a segment *AB* in ratio 3: 1.

f) the length of *AL* — a median of the triangle *ABC*;

g) the equation of a straight line *AM*, which forms an angle 45^0 with a straight line *AB*.

2. The points *A(– 3, 4), B(5, 1), C(4, – 3)* are vertices of a parallelogram *ABCD*. Find:

a) the equation of a straight line *AB*;

b) the equation of a straight line *CD*;

c) the coordinates of a point *D*;

d) the equation *AN* — a median of a triangle *ABC*;

e) the equation *AK* — altitude of a triangle *ABC*;

f) the area of a parallelogram *ABCD*;

g) the area of a triangle *ABM*, where a point *M* is a point of intersection of medians of the triangle *ABC*.

3. Write the equation of the line, which is
a) parallel; b) perpendicular to the line $-3x + 7y = 11$, and passes through the point $(2, 4)$.

4. Write the equation of the line, which passes through the points: a) $(-1, 2)$ and $(5, 2)$; b) $(-2, -3, 1)$ and $(4, 5, 6)$;
c) $(1, -1, 2, 0)$ and $(-7, 2, -9, 1)$.

5. What is the difference between the lines l_1 and l_2, if

$$l_1 : \frac{x-2}{4} = \frac{y+4}{-2} = \frac{z-7}{-8}; \qquad l_2 : \frac{x-2}{-2} = \frac{y+4}{1} = \frac{z-7}{4}.$$

6. Write the equation of the line, which is parallel to the line $\frac{x-1}{-2} = \frac{y+1}{3} = \frac{z-5}{6}$ and passes through the point $(7,-2,15)$.

7. Official exchange rate of the Russian ruble to the US dollar was 3 623:1 on January 5, 1995 and 4 133:1 on February 6, 1995. What was the approximate exchange rate on January 17, considering that the rate was changing linearly?

Appendix

Having deciphered an ancient recording, scientists determined that the "Bagynbas" stronghold was constructed in the form of rhombus, one of the vertices of which was located in 5 stadia (stadium ['stādēəm] sta·di·um (pl. – dia) an ancient Roman or Greek measure of length, about 185 meters) *to the north and 5 stadia to the east from the big well. The center of the stronghold was located in 1 stadium to the north and 2 stadia to the east from the big well. According to these data scientists determined the location of another vertex and the equation of the line which had the rest of the vertices. Then, as the result of the additional research, they determined that the length of the wall was 13 stadia.*

Help the scientists to determine the coordinates of the other vertices and the area of the stronghold.

Let's take the big well as the point of the coordinate origin:

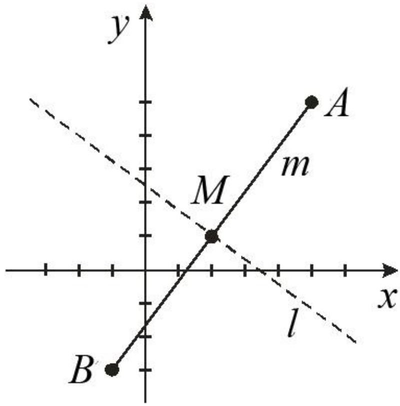


Figure 5.2

Vertex *A* and the center of the stronghold *M* are located on a straight line *m*, equation to which can be written in many ways. In particular, in order to write it in the form of linear function, we write it using indefinite coefficients $y = kx + b$, then substituting coordinates of the points *A* and *M*, we get the system, which determines the values of the coefficients:

$$\begin{cases} 5 = k \cdot 5 + b; \\ 1 = k \cdot 2 + b. \end{cases}$$

Having solved the system we write the equation of the straight line *m*: $y = (4/3)x + (-5/3)$.

Length of the line segment *MA* equals

$$|MA| = \sqrt{(5-2)^2 + (5-1)^2} = \sqrt{25} = 5.$$

Then, knowing that the vertex *B* is located on the straight line *m* and the length of the line segment *BM* equals to the line segment *MA*, we get the system of equations in order to determine the coordinates of *B*:

$$\begin{cases} (x_B - 2)^2 + (y_B - 1)^2 = 5^2; \\ y_B = \frac{4}{3}x_B - \frac{5}{3}. \end{cases} \quad (5.12)$$

In order to solve the system (5.12) we substitute the value of y_B into the first equation of system (5.12), and then we remove the brackets, combine the like terms and solve the obtained quadratic equation. In our case, it is easy to solve if we notice, that we can factor out the common multiplier from the second brackets, then we get the equation:
$(x_B - 2)^2 + (16/9)(x_B - 2)^2 = 25$. Then, $(25/9)(x_B - 2)^2 = 25$, and $x_B = 7$ or $x_B = -1$. As opposed to the previous problem, one of the roots ($x_B = 7$) is irrelevant. So, vertex B has the following coordinates $(-1, -3)$.
Straight line l, which has two remaining vertices, is perpendicular to m. Therefore, slope is "twice mutually inverse" to the slope of the straight line l:
$k_l = (-1/k_m) = (-1/(4/3)) = -3/4$.
Then the equation of the line l, which goes through the point M: $y - 1 = (-3/4)(x - 2)$ or $y = (-3/4)x + 2.5$.
The rest of the vertices of the rhombus are located on the line l, at a distance of 13 stadia from the vertex A:

$$\begin{cases} (x - 5)^2 + (y - 5)^2 = 13^2; \\ y = -\frac{3}{4}x + \frac{5}{2}. \end{cases}$$

Substituting the value of y from the 2nd equation of the system into the 1st equation, then having removed brackets, we combine the like terms and solve the obtained quadratic equation: $25x^2 - 100x - 2\,204 = 0 \Rightarrow x_1 = 11.6;\ x_2 = -7.6$. Consequently, coordinates of the two remaining vertices of the rhombus are $(11.6, -6.2)$ and $(-7.6, 8.2)$.
The same answer can be obtained without having to write the equation of the line l, if we notice that the vertices of the

rhombus are the cross points of two circles with the radius of 13 and with the centers A and B, and if we solve the corresponding system: $\begin{cases} (x-5)^2 + (y-5)^2 = 13^2; \\ (x+1)^2 + (y+3)^2 = 13^2. \end{cases}$

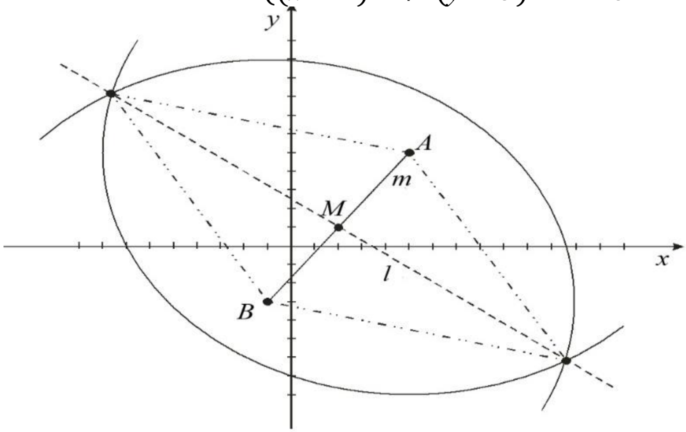


Figure 5.3

§6. Linear Equations

6.1. Problem

Price of the shirt is 250 soms. Average variable costs for production of one shirt (cost of fabrics, accessories, salaries, etc.) are 170 soms. How many shirts is it necessary to produce and sell, in order to recover the costs, if fixed costs (rent, equipment, obtaining permits, etc.) are 100 000 soms?

Solution

Denoting the number of shirts through q, we find that the revenue from the sale of shirts R is equal to $250q$, and the total costs $C = 170q + 100\ 000$. The costs are covered when the revenue is equal to the costs: $R = C$. As a result, we have the equation: $250q = 170q + 100\ 000.$ Its' solution, $q = 12\ 500.$

Let's note that this number is usually named a *break-even point*.

The equation $250q = 170q + 100\ 000,$ which we solved, is named *a linear equation*.

Definition

Two equations are called ***equivalent***, if the roots of the first equation, and only they, are the roots of the second. The two equations are equivalent, if one of them can lead to another using just elementary transformations, i.e.:

1. Multiplying (dividing) the equation on the number not equal to zero;

2. Adding (deducting) the same expression to the both sides of equation.

Definition

The equation is called as a linear algebraic equation with n unknowns, if using elementary transformations it comes to the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b. \quad (6.1)$$

The numbers $a_1, a_2, \dots a_n$ are called the coefficients of the equation, the number b is right-hand side or the free term of the equation. The left side of equation (6.1) is called a linear function of n variables.

Solving linear equations, we can have three different situations. The equation can:

I. not have roots, if using elementary transformations, it comes to the equation $0 \cdot x = b$ $(b \neq 0)$. Put any number instead of x on the left side, we get zero, while the right side is not zero.

II. have one and only one root, if it comes to the form $a \cdot x = b$ $(a \neq 0)$. Then $x = b/a$.

III. have infinitely many roots, if it comes to the form $0 \cdot x = 0$ (then for any value of x, there is an equity $0 \cdot x = 0$), or to the form $ax_k + ... + cx_m = b$ $(k \neq m, a \neq 0, c \neq 0)$. In this case, all unknowns, except x_k , can be chosen arbitrarily, and x_k can be expressed through them.

Problem *Solve the equations:*

1) $15(2x - 1) = 8(3x + 4) - 3(9 - 2x);$
2) $5(14x + 21) - 33 = 7(4x - 11) + 3x + 71;$
3) $52(3x - 21) - 2(7x + 4) = 17(5x + 2) + 19(3x - 60) + 6;$
4) $5x_1 - 7x_2 = 34.$

Solution

1) We are opening the brackets and combining like terms:
$15(2x - 1) = 8(3x + 4) - 3(9 - 2x) \Leftrightarrow$
$\Leftrightarrow 30x - 15 = 24x + 32 - 27 + 6x \Leftrightarrow 30x - 15 = 30x + 5 \Leftrightarrow$
$\Leftrightarrow 0 \cdot x = 20$. The resulting equation has no roots.

2) We are opening the brackets and combining like terms:
$5(14x + 21) - 33 = 7(4x - 11) + 3x + 71 \Leftrightarrow$
$\Leftrightarrow 70x + 72 = 31x - 6 \Leftrightarrow 39x = -78 \Leftrightarrow x = -2.$

3) We are opening the brackets and combining like terms:
$52(3x - 21) - 2(7x + 4) = 17(5x + 2) + 19(3x - 60) + 6 \Leftrightarrow$
$\Leftrightarrow 156x - 1092 - 14x - 8 = 85x + 34 + 57x - 1140 + 6 \Leftrightarrow$
$\Leftrightarrow 42x - 1100 = 142x - 1100 \Leftrightarrow 0 \cdot x = 0$. The resulting equation has infinitely many roots.

4) The solution $x_1 = 4$, $x_2 = -2$ or shorter $(4; -2)$ of the equation $5x_1 - 7x_2 = 34$ can be obtained by setting $x_2 = -2$ and solving equation $5x_1 + 14 = 34$. If we set $x_2 = 1$ we obtain the solution $(8.2; 1)$. It is clear that we can set x_2 infinitely many values and, accordingly, receive infinitely many solutions. To write these solutions let we set $x_2 = a$, where a is any number. Mathematicians usually said, a is parameter.

Then, $5x_1 - 7a = 34 \Leftrightarrow 5x_1 = 34 + 7a \Leftrightarrow x_1 = (34+7a)/5.$

As result, all solutions of the equation $5x_1 - 7x_2 = 34$ can be written in a parametric form: $((34+7a)/5;\ a)$, $a \in R$.

Exercises 6.1

Solve the equations:

C. 1) $11(4x - 3) - 17x = 84 + 3(9x - 22)$;
2) $6x + 17 = 5(4x - 11) - (4 - x) + 7$;
3) $2(13x - 1) - 8(7 - 4x) = 19 - 7(5x + 2) + 3(31x - 6) - 45$;
4) $8x - 3y + 7z = 3x + 32$.

H. 1) $9(5 - 2x) = 13 - 3(6x + 19)$; 2) $7x + 72 = 5x$;
3) $21(3x - 4) - 2(5x + 4) = 53(x + 2) - 14$;
4) $4x + 11y = 23$.

6.2. Problem

Examine an equation to be solved for x and in those cases when solutions exist, find them:
1) $4 - 2x = 21a$; 2) $11x + 37 = cx + 8$;
3) $7(3x + 2) - d(6 - 4x) = 2e + 15$.

Solution

1) We can rewrite the equation $4 - 2x = 21a$ in the form $2x = 4 - 21a$. This equation has a unique solution, because in this equation the coefficient for the unknown is non–zero (equal to 2): $x = (4 - 21a) / 2 = 2 - 10.5a$.

2) Combining like terms in the equation $11x + 37 = cx + 8$, we obtain $(11 - c)x = -29$.

This equation has a unique solution $x = -29/(11 - c)$, when the coefficient for the unknown is non–zero: $11 - c \neq 0$, that is, when $c \neq 11$.

In the case when $11 - c = 0 \Leftrightarrow c = 11$, the equation $(11 - c)x = -29$ has no roots.

3) Open the brackets and combine like terms:

$7(3x + 2) - d(6 - 4x) = 2e + 15 \Leftrightarrow 21x + 14 - 6d + 4dx =$
$= 2e + 15 \Leftrightarrow (21 + 4d)x = 2e + 6d + 1$.

The equation $(21 + 4d)x = 2e + 6d + 1$ has a unique solution $x = (2e + 6d + 1)/(21 + 4d)$ when the coefficient for the unknown is non-zero: $21 + 4d \neq 0$, that is, when $d \neq -5.25$.

If $21 + 4d = 0 \Leftrightarrow d = -5.25$, two cases are possible:

– the equation $(21 + 4d)x = 2e + 6d + 1$ has no roots when $d = -5.25$ and $e \neq 15.25$. In this case, the equation will take the form $0x = 2e - 30.5$, with the right side non-zero.

– the equation $(21 + 4d)x = 2e + 6d + 1$ has infinitely many roots when $d = -5.25$ and $e = 15.25$. In this case, the equation takes the form $0 \cdot x = 0$, and the solution can be written in the form $x = p$, where p is any number.

Exercises 6.2

Examine an equation to be solved for x and in those cases when solutions exist, find them:

C. 1) $4x + 2a = 17a - 87 + 7x$; 2) $11 - 9x + 3c = cx + 18$;
3) $5dx + 4 = 2 - d(x + 3) - 12e$.

H. 1) $12x - 7 = 21 + 3a$; 2) $19 - 2x = 5cx + 7$;
3) $7x + 36 - 42d = (56 - x)e + 15$.

6.3. Problem

Examine a system of equations to be solved for x and y, and in those cases when solutions exist, find them:

1) $\begin{cases} 13(x - a) + 7y = 8x + 11(y + 2); \\ 3x + 8a + 17y = 6x + 3(2y - x + 10a); \end{cases}$

2) $\begin{cases} 4(x + 3b) + 7(1 - 2y) = bx + 14(2 - y); \\ 3x + 7y = 6x + 32y + 10b; \end{cases}$

3) $\begin{cases} 3(cx - a) + y = cx + 2(3y + cx); \\ x + 2a + 3y = dx + 4(5y - 6x + 7a). \end{cases}$

Solution

1) Open the brackets and combine like terms:

$$\begin{cases} 13(x - a) + 7y = 8x + 11(y + 2); \\ 3x + 8a + 17y = 6x + 3(2y - x + 10a); \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 13x - 13a + 7y = 8x + 11y + 22; \\ 3x + 8a + 17y = 6x + 6y - 3x + 30a; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 5x - 4y = 13a + 22; \\ 11y = 22a. \end{cases}$$

The result is a degenerate system. Therefore, we first calculate the value of y from the second equation: $11y = 22a \Leftrightarrow y = 2a$. Then, substitute the found value of y in the first equation and calculate x: $5x - 4y = 13a + 22 \Leftrightarrow$ $\Leftrightarrow 5x - 4 \cdot 2a = 13a + 22 \Leftrightarrow 5x = 21a + 22 \Leftrightarrow$ $\Leftrightarrow x = 4.2a + 4.4.$

Answer $(4.2a + 4.4; 2a)$, where a is any number.

2) Opening the brackets and combining like terms we have a degenerate system:

$$\begin{cases} 4(x + 3b) + 7(1 - 2y) = bx + 14(2 - y); \\ 3x + 7y = 6x + 32y + 10b; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 4x + 12b + 7 - 14y = bx + 28 - 14y; \\ 3x + 7y = 6x + 32y + 10b; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (4 - b)x = 21 - 12b; \\ 3x + 25y = -10b. \end{cases}$$

The first equation of the system, and therefore, the system itself has no solution if $b = 4$.

In the case when $b \neq 4$, we get $x = (21 - 12b)/(4 - b)$.

Then, $3(21 - 12b)/(4 - b) + 25y = -10b \Leftrightarrow$

$\Leftrightarrow y = (10b^2 - 4b - 63)/(100 - 25b)$.

3) Opening the brackets and combining like terms we have a degenerate system:

$$\begin{cases} 3(cx - a) + y = cx + 2(3y + cx); \\ x + 2a + 3y = dx + 4(5y - 6x + 7a), \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3cx - 3a + y = cx + 6y + 2cx; \\ x + 2a + 3y = dx + 20y - 24x + 28a, \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 5y = -3a; \\ (25 - d)x - 17y = 26a, \end{cases} \Leftrightarrow \begin{cases} y = -0.6a; \\ (25 - d)x = 15.8a. \end{cases}$$

Answer: If $d \neq 25$ system has an unique solution $(15.8a/(25 - d); - 0.6a)$, where a is any number;

if $a = 0$; $d = 25$ system has infinitely many solutions $(p; 0)$, where p is any number;
if $a \neq 0$; $d = 25$ system has no solution.

Exercises 6.3.

Examine a system of equations to be solved for x and y, and in those cases when solutions exist, find them:

C. 1) $\begin{cases} 15x - 21a + 4y = 8x + 7(x + 2y + 3); \\ 8x + 7y = 6x + 5(4y - 3 + 2a); \end{cases}$

2) $\begin{cases} 4 - 5x + 3b + 2(1 - 2y) = bx + 14 - y; \\ 2(3x - 7y) = 6x + 2by + 10b; \end{cases}$

3) $\begin{cases} 3cx - d + 2y = cx + 3y; \\ 9x - 2y + 3d = 4x + 5(x - 2y + 3d). \end{cases}$

H. 1) $\begin{cases} 11x + 6y = 8x - 3(4y - x + 2a); \\ 3a - 2x + 19y = 6x + 2(2y - 9a); \end{cases}$

2) $\begin{cases} 4x + 3y - 2b = 2x + b(2 - y); \\ 13x + 6y = 6x + 3(2y + 7b); \end{cases}$

3) $\begin{cases} 2cx + 5y = 6cx + 2(3c - 5y + x); \\ 5x + 2y + 3d = x - 4(6 - x - 3y). \end{cases}$

6.4. Theorem

In the previous section, we discussed degenerate systems of linear equations. It is time to move on to general systems. It may seem like it will be much more complicated. But fortunately, by Cramer's theorem, everything is ready for us.

So, we consider a system of two linear equations of two variables in a general form:

$$\begin{cases} a_{11}x + a_{12}y = b_1, \\ a_{21}x + a_{22}y = b_2. \end{cases} \qquad (6.2)$$

We assume that all the coefficients a_{ij} are nonzero. Otherwise, a degenerate system takes place and we are able to handle them.

Theorem

System (6.2) is equivalent to each of the following systems:

$$\begin{cases} a_{11}x + a_{12}y = b_1; \\ \Delta \cdot x = \Delta_x; \end{cases} \qquad \begin{cases} a_{11}x + a_{12}y = b_1; \\ \Delta \cdot y = \Delta_y; \end{cases}$$

$$\begin{cases} a_{21}x + a_{22}y = b_2; \\ \Delta \cdot x = \Delta_x; \end{cases} \qquad \begin{cases} a_{21}x + a_{22}y = b_2; \\ \Delta \cdot y = \Delta_y. \end{cases}$$

Proof

The validity of the theorem directly follows from Cramer's theorem.

Problem *Examine a system of equations to be solved for x and y, and in those cases when solutions exist, find them:*

1) $\begin{cases} 12x - 7y = a; \\ 2x + 3y = 16; \end{cases}$ 2) $\begin{cases} 2x + 3y = 7; \\ 5x + ay = 13; \end{cases}$

3) $\begin{cases} 3x - (c+5)y = d + 2; \\ cx + 2y = 7. \end{cases}$

Solution

1) We begin the solution of the problem by calculating the corresponding determinants:

$$\Delta = \begin{vmatrix} 12 & -7 \\ 2 & 3 \end{vmatrix} = 12 \cdot 3 - 2 \cdot (-7) = 36 + 14 = 50;$$

$$\Delta_x = \begin{vmatrix} a & -7 \\ 16 & 3 \end{vmatrix} = a \cdot 3 - 16 \cdot (-7) = 3a + 112.$$

According to the theorem, we obtain

$$\begin{cases} 12x - 7y = a; \\ 2x + 3y = 16; \end{cases} \Leftrightarrow \begin{cases} \Delta \cdot x = \Delta_x; \\ 2x + 3y = 16; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 50x = 3a + 112; \\ 2x + 3y = 16; \end{cases} \Leftrightarrow \begin{cases} x = 0.06a + 2.24; \\ 2(0.06a + 2.24) + 3y = 16; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 0.06a + 2.24; \\ 3y = 11.52 - 0.12a; \end{cases} \Leftrightarrow \begin{cases} x = 0.06a + 2.24; \\ y = 3.84 - 0.04a. \end{cases}$$

2) We begin the solution of the problem by calculating the corresponding determinants:

$$\Delta = \begin{vmatrix} 2 & 3 \\ 5 & a \end{vmatrix} = 2 \cdot a - 5 \cdot 3 = 2a - 15;$$

$$\Delta_y = \begin{vmatrix} 2 & 7 \\ 5 & 13 \end{vmatrix} = 2 \cdot 13 - 5 \cdot 7 = -9.$$

According to the theorem, we obtain

$$\begin{cases} 2x + 3y = 7; \\ 5x + ay = 13; \end{cases} \Leftrightarrow \begin{cases} \Delta \cdot y = \Delta_y; \\ 2x + 3y = 7; \end{cases} \Leftrightarrow \begin{cases} (2a - 15)y = -9; \\ 2x + 3y = 7. \end{cases}$$

If $2a - 15 = 0 \Leftrightarrow a = 7.5$, the system has no solution.

If $2a - 15 \neq 0 \Leftrightarrow a \neq 7.5$, we obtain

$$\begin{cases} (2a - 15)y = -9; \\ 2x + 3y = 7; \end{cases} \Leftrightarrow \begin{cases} y = -9/(2a - 15); \\ 2x + 3[-9/(2a - 15)] = 7; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = -9/(2a - 15); \\ x = (7a - 39)/(2a - 15). \end{cases}$$

3) As usual we begin the solution of the problem by calculating the corresponding determinants:

$$\Delta = \begin{vmatrix} 3 & -(c + 5) \\ c & 2 \end{vmatrix} = 3\cdot 2 - c\cdot[-(c + 5)] = c^2 + 5c + 6;$$

$$\Delta_y = \begin{vmatrix} 3 & d + 2 \\ c & 7 \end{vmatrix} = 21 - c(d + 2).$$

According to the theorem, we obtain

$$\begin{cases} 3x - (c + 5)y = d + 2; \\ cx + 2y = 7; \end{cases} \Leftrightarrow \begin{cases} \Delta \cdot y = \Delta_y; \\ cx + 2y = 7; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (c^2 + 5c + 6)y = d + 2; \\ cx + 2y = 7. \end{cases}$$

If $c^2 + 5c + 6 = 0 \Leftrightarrow (c + 2)(c + 3) = 0 \Leftrightarrow c = -2$ or $c = -3$ and $d + 2 \neq 0 \Leftrightarrow d \neq -2$ the system has no solution.

If $c^2 + 5c + 6 = 0 \Leftrightarrow (c + 2)(c + 3) = 0 \Leftrightarrow c = -2$ or $c = -3$ and $d + 2 = 0 \Leftrightarrow d = -2$ the system has infinitely many solutions:

$$\begin{cases} 0 \cdot y = 0; \\ cx + 2y = 7; \end{cases} \Leftrightarrow \begin{cases} y = p; \\ cx + 2p = 7; \end{cases} \Leftrightarrow \begin{cases} y = p; \\ x = (7 - 2p)/c, \end{cases}$$

where p is any number.

If $c^2 + 5c + 6 \neq 0 \Leftrightarrow (c + 2)(c + 3) \neq 0 \Leftrightarrow c \neq -2$ and $c \neq -3$ the system have unique solution:

$$\begin{cases} y = (d + 2)/(c^2 + 5c + 6); \\ x = (7 - 2y)/c; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = (d + 2)/(c^2 + 5c + 6); \\ x = [7 - 2(d + 2)/(c^2 + 5c + 6)]/c. \end{cases}$$

Note that the coefficient ***c*** is assumed to be nonzero according to the condition on the coefficients of the system.

Exercises 6.4.

Examine a system of equations to be solved for *x* and *y*, and in those cases when solutions exist, find them:

C. 1) $\begin{cases} 2x - 7y = 5 + a; \\ x + 9y = 6; \end{cases}$ 2) $\begin{cases} 2x + (3 - b)y = 5; \\ 5x + 4y = 9; \end{cases}$

3) $\begin{cases} 11x + (3 - c)y = 14; \\ 5x - 2y = 3 + d. \end{cases}$

H. 1) $\begin{cases} 4x - y = 7; \\ 13x + 3y = a; \end{cases}$ 2) $\begin{cases} 5x + 3y = 4; \\ 8x - ay = 3; \end{cases}$

3) $\begin{cases} 3x - cy = 2; \\ (5 - c)x - 2y = c + 8. \end{cases}$

6.5. Graphic interpretation: a unique solution

As noted in §5, an equation of the form $Ax + By = C$ is an equation of the straight line. Accordingly, two equations of this kind describe two straight lines, and the solution of the system of two equations is the coordinates of the point of intersection of the straight lines.

Problem

Solve system of linear equations and draw graph of equations: $\begin{cases} 12x - 7y = 84; \\ 2x + 3y = -11. \end{cases}$

Solution

We begin the solution of the problem by calculating the determinant of coefficients' matrix:

$$\Delta = \begin{vmatrix} 12 & -7 \\ 2 & 3 \end{vmatrix} = 12 \cdot 3 - 2 \cdot (-7) = 50$$

and the determinant Δ_x :

$$\Delta_x = \begin{vmatrix} 84 & -7 \\ -11 & 3 \end{vmatrix} = 84 \cdot 3 - (-11) \cdot (-7) = 175.$$

Therefore, the initial system is equivalent to the system:

$$\begin{cases} 12x - 7y = 84; \\ 2x + 3y = -11; \end{cases} \Leftrightarrow \begin{cases} \Delta \cdot x = \Delta_x; \\ 2x + 3y = -11; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 50x = 175; \\ 2x + 3y = -11; \end{cases} \Leftrightarrow \begin{cases} x = 3.5; \\ 2\cdot 3.5 + 3y = -11; \end{cases} \Leftrightarrow \begin{cases} x = 3.5; \\ y = -6. \end{cases}$$

In order to draw graphs of these equations, it is convenient to rewrite them in the form of equations in segments: $\begin{cases} 12x - 7y = 84; \\ 2x + 3y = -11; \end{cases} \Leftrightarrow \begin{cases} x/7 + y/(-12) = 1; \\ x/(-5.5) + y/(-11/3) = 1. \end{cases}$

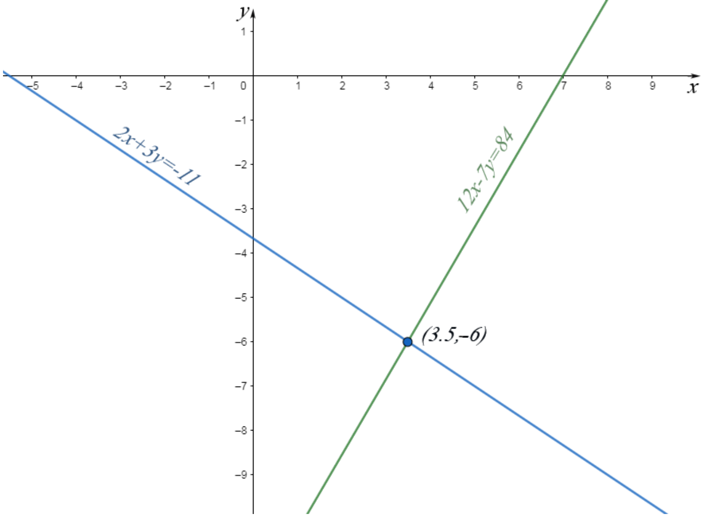


Figure 6.1

It is easy to see that the coordinates of the point of intersection of the straight lines are the solution of the system.

Exercises 6.5.

Solve system of linear equations and draw a graph of equations:

C. $\begin{cases} 11x + 8y = 88; \\ 5x + 4y = 42. \end{cases}$ **H.** $\begin{cases} 4x - y = 0.2; \\ 13x + 5y = 65. \end{cases}$

6.6. Graphic interpretation: no solutions

When discussing the properties of the equation of the straight line in §5, it was noted that the coefficients A and B of the equation $Ax + By = C$ can be considered as the coordinates of a vector, which perpendicular to the straight line described by this equation. It follows that if the coefficients of the equations of the system are proportional, that is, the perpendicular vectors are parallel, the equations of the system describe parallel lines. Such straight lines do not have intersection points. Let us see how this is expressed in the language of systems of linear equations.

Problem

Solve system of linear equations and draw graph of equations: $\begin{cases} 12x - 7y = 42; \\ -24x + 14y = -168. \end{cases}$

Solution

We begin the solution of the problem by calculating the determinant of coefficients' matrix:

$$\Delta = \begin{vmatrix} 12 & -7 \\ -24 & 14 \end{vmatrix} = 12\cdot 14 - (-24)\cdot(-7) = 0$$

and the determinant Δ_x :

$$\Delta_x = \begin{vmatrix} 42 & -7 \\ -168 & 14 \end{vmatrix} = 42\cdot 14 - (-168)\cdot(-7) = 588 - 1176 = -588.$$

Therefore, the initial system is equivalent to the system:

$$\begin{cases} 12x - 7y = 42; \\ -24x + 14y = -168; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \Delta \cdot x = \Delta_x; \\ -24x + 14y = -168; \end{cases} \Leftrightarrow \begin{cases} 0 \cdot x = -588; \\ -24x + 14y = -168. \end{cases}$$

This system has no solution.

In order to draw graphs of these equations, it is convenient to rewrite them in the form of equations in segments: $\begin{cases} 12x - 7y = 42; \\ -24x + 14y = -168; \end{cases} \Leftrightarrow \begin{cases} x/3.5 + y/(-6) = 1; \\ x/7 + y/(-12) = 1. \end{cases}$

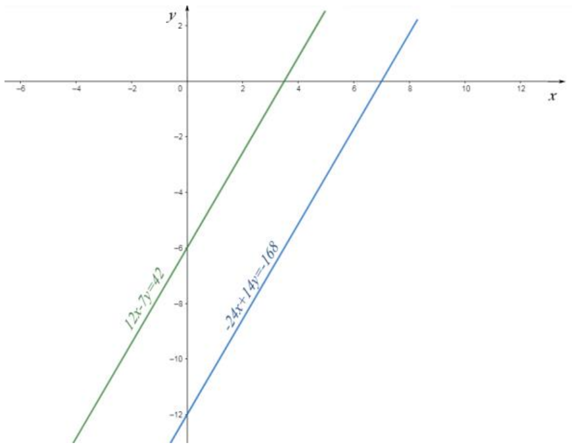


Figure 6.2

So, it turned out that parallel and, therefore, disjoint straight lines corresponds to a system that has no solutions, for which $\Delta = 0$ and $\Delta_x \neq 0$.

Exercises 6.6.

Solve system of linear equations and draw graph of equations:

C. $\begin{cases} 10x + 8y = 88; \\ 5x + 4y = 40. \end{cases}$ **H**. $\begin{cases} 4x - y = 2; \\ -12x + 3y = -36. \end{cases}$

6.7. Graphic interpretation: infinitely many solutions

Two lines have infinitely many intersection points only if these lines coincide. In the language of equations, this means that the coefficients of one equation of the system are obtained by multiplying the corresponding coefficients of another equation by the same number.

Problem

Solve system of linear equations and draw a graph of equations: $\begin{cases} 12x - 16y = 48; \\ -3x + 4y = -12. \end{cases}$

Solution

Let calculate the determinant of coefficients' matrix: $\Delta = \begin{vmatrix} 12 & -16 \\ -3 & 4 \end{vmatrix} = 12 \cdot 4 - (-3) \cdot (-16) = 0$

and the determinant Δ_x :

$$\Delta_x = \begin{vmatrix} 48 & -16 \\ -12 & 4 \end{vmatrix} = 48 \cdot 4 - (-12) \cdot (-16) = 192 - 192 = 0.$$

Note that the determinant Δ_y is also equal to zero:

$$\Delta_y = \begin{vmatrix} 12 & 48 \\ -3 & -12 \end{vmatrix} = 12 \cdot (-12) - 48 \cdot (-3) = -144 + 144 = 0.$$

Therefore, the initial system is equivalent to the system:

$$\begin{cases} 12x - 16y = 48; \\ -3x + 4y = -12; \end{cases} \Leftrightarrow \begin{cases} \Delta \cdot x = \Delta_x; \\ -3x + 4y = -12; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 0 \cdot x = 0; \\ -3x + 4y = -12. \end{cases}$$

This system has infinitely many solutions $(p;\ 0.75p - 3)$, where p is any number.

In order to draw graphs of these equations, it is convenient to rewrite them in the form of equations in segments: $\begin{cases} 12x - 16y = 48; \\ -3x + 4y = -12; \end{cases} \Leftrightarrow \begin{cases} x/4 + y/(-3) = 1; \\ x/4 + y/(-3) = 1. \end{cases}$

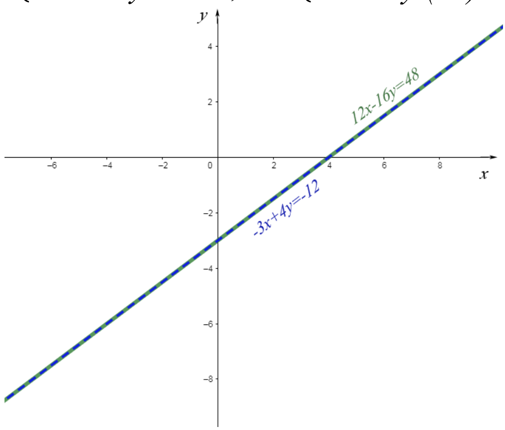


Figure 6.3

Exercises 6.7.

Solve system of linear equations and draw a graph of equations:

C. $\begin{cases} 15x + 6y = 90; \\ 5x + 2y = 30. \end{cases}$ **H**. $\begin{cases} 4x - 7y = 28; \\ -12x + 21y = -84. \end{cases}$

Comment

Summarizing, we can say that the system $\begin{cases} a_{11}x + a_{12}y = b_1, \\ a_{21}x + a_{22}y = b_2 \end{cases}$ – has an unique solution, if $\Delta \neq 0$. This situation is illustrated by two straight lines that intersect at a point whose coordinates are a solution to the system;

– has no solution, if $\Delta = 0$, and at least one of the determinants Δ_x, Δ_y is not 0. This situation is illustrated by two parallel straight lines.

– has infinitely many solutions, if $\Delta = 0$ and $\Delta_x = \Delta_y = 0$. This situation is illustrated by two coincide straight lines.

6.8. Problem

a) A seamstress can sew a sundress in 25 minutes and a dress in 1 hour. An apprentice seamstress a week after the start of training could sew a sundress in 1 hour 30 minutes and a dress in 2 hours. They sewed the same number of sundresses and dresses. The seamstress worked for 4 hours 30 minutes, and the apprentice seamstress worked for 13 hours. How many sundresses and dresses did each sew?

b) A week later, having measured the time, Adyl said that the apprentice seamstress can sew a sundress in 45 minutes and a dress in 1 hour 48 minutes. Adyl also said that they sewed the same number of sundresses and dresses when the seamstress worked 4.2 hours, and the apprentice seamstress worked for 11 hours 15 minutes. How many sundresses and dresses did each sew?

c) After it turned out that the last task had no solution, they began to look for an error and found out that the seamstress had spent not 4.2 hours, but 6 hours 25 minutes.

Solve this problem. Illustrate each situation by drawing graphs of the corresponding equations.

Solution

Denote by x the number of sundresses, by y — dresses. In order not to mess with fractional numbers, we will translate all the data in minutes, write and solve the corresponding systems.

a) $\begin{cases} 25x + 60y = 270; \\ 90x + 120y = 780. \end{cases}$

As usually, we begin the solution of the problem by calculating the determinant of coefficients' matrix: $\Delta = \begin{vmatrix} 25 & 60 \\ 90 & 120 \end{vmatrix} = -2\,400$ and the determinant Δ_x :

$\Delta_x = \begin{vmatrix} 270 & 60 \\ 780 & 120 \end{vmatrix} = -14\,400.$

Then $x = -14\,400/(-2\,400) = 6$ and, therefore, $y = 2$.

In order to draw graphs of these equations, it is convenient to rewrite them in the form of equations in segments:

$$\begin{cases} 25x + 60y = 270; \\ 90x + 120y = 780; \end{cases} \Leftrightarrow \begin{cases} x/10.8 + y/4.5 = 1; \\ \dfrac{x}{26/3} + \dfrac{y}{6.5} = 1. \end{cases}$$

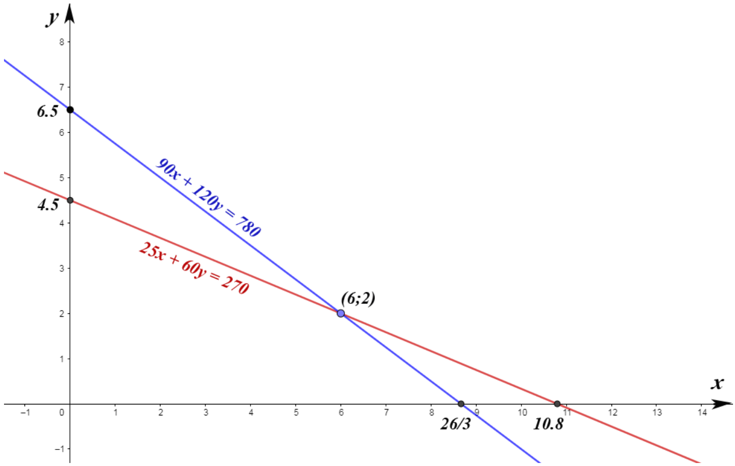


Figure 6.4

b) $\begin{cases} 25x + 60y = 252; \\ 45x + 108y = 675. \end{cases}$ We calculate the determinant of coefficients' matrix: $\Delta = \begin{vmatrix} 25 & 60 \\ 45 & 108 \end{vmatrix} = 25 \cdot 108 - 45 \cdot 60 = 0$ and the determinant

$$\Delta_x = \begin{vmatrix} 252 & 60 \\ 675 & 108 \end{vmatrix} = 252 \cdot 108 - 675 \cdot 60 = 27\,216 - 40\,500 \neq 0.$$

It turned out that the system has no solution. Apparently, there are errors in the problem's data.

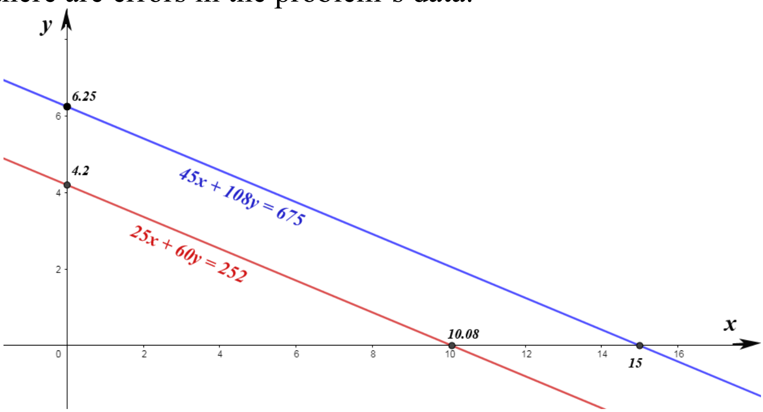


Figure 6.5

c) $\begin{cases} 25x + 60y = 375; \\ 45x + 108y = 675. \end{cases}$ We know that the determinant of coefficients' matrix equals zero.

The determinants:

$\Delta_x = 375\cdot108 - 675\cdot60 = 40\,500 - 40\,500 = 0,$

$\Delta_y = 375\cdot45 - 675\cdot5 = 16\,875 - 16\,875 = 0.$

This situation occurs when the lines defined by the equations of the system coincide.

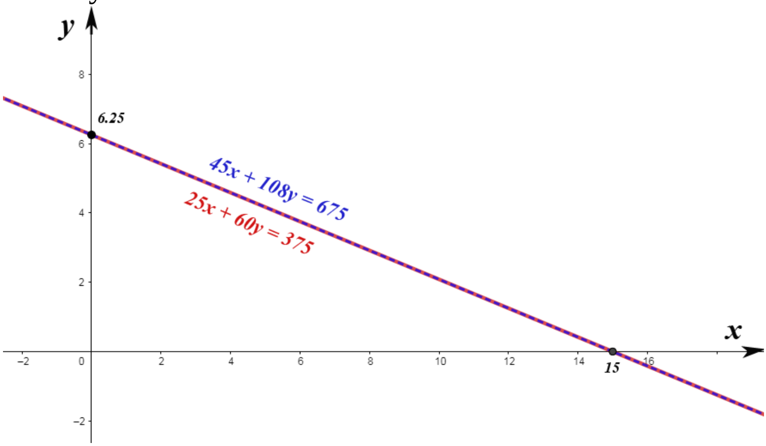


Figure 6.6

So, the initial system is equivalent to the system:

$$\begin{cases} x = p; \\ 45x + 108y = 675; \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = p; \\ y = (675 - 45p)/108; \end{cases} \Leftrightarrow \begin{cases} x = p; \\ y = (75 - 5p)/12. \end{cases}$$

According to the conditions of the problem, the variables x and y must be integer and non-negative. Since $x = p \geq 0$ and $y = (75 - 5p)/12 \geq 0$, we get $0 \leq p \leq 15$. Further, from the integer condition, we obtain that from $p = 0, 1, \ldots, 15$, the variable y will be integer only at $p = 3$ or $p = 15$. So, it turned out that the solutions are $x = 3; \ y = 5$ or $x = 15; \ y = 0$.

Exercises 6.8.

Determine a number of system's solutions and draw graph of equations:

C. 1) $\begin{cases} 5x - 20y = -21; \\ 62x - 248y = -260.4. \end{cases}$ 2) $\begin{cases} 6x - 24y = -25.2; \\ -31x + 124y = 132.2. \end{cases}$

3) $\begin{cases} -9x + 36y = 37.8; \\ 93x - 225y = -388.5. \end{cases}$

H. 1) $\begin{cases} 24x + 75y = 195; \\ 16x + 50y = 130. \end{cases}$ 2) $\begin{cases} 12x + 37.5y = 99.5; \\ 4.8x + 15y = 39. \end{cases}$

3) $\begin{cases} 8x + 25y = 65; \\ 9.6x + 30y = 78. \end{cases}$

Summary

1. Solve the equations:

a) $2x - 7 = 0;$ b) $3x - 2 = 7 - 5x;$

c) $0.2x - 5.5 = x/3 - 4.1;$ d) $5(x - 2) = 7(3x + 2) - 9;$

e) $3x = 15a;$ f) $3(5 - 2x)/2 + 4x = 1$

2. Examine an equation to be solved for x and in those cases when solutions exist, find them:

a) $4a - 5x = -2a$; b) $7x + 3d = 2x$; c) $3cx + 5 = 2$;

d) $2cx - d = 5d$; e) $4cx - 2d = 5n$;

f) $5(3a - x) - 4(2a - 7x) = -3x;$ g) $3(5 + 2x) - a(7 - x) = b.$

3. Solve the equation and write the answer in parametric form:

a) $2x - 5y = 8$; b) $3x - 8y = 12 - 5y$; c) $2(x + y) = 6x$;

d) $2x - 3y + 5z = 13$; e) $2x + 2(7 - 19y) = 3z$;

f) $x + 2y + 3z - 4t = 7;$ g) $x_1 + 2x_2 + \ldots + nx_n = n.$

4. If to a number is added it's three times value, then 88 will be obtained. Find the number.

5. A seller sold a suit for $140 and made the profit equal to 3/4 costs. Find the amount of cost and the amount of profit, knowing that the price is equal to the sum of profit and cost.

6. A father is four times older than his daughter at and 27

years older than she is. How old is the father?
7. A shop has two sorts of chocolates; one is for \$ 3.34 per a kilogram, the second – for \$2.8. How many kilograms of each sort of chocolates should be used to obtain 90 kg of a mixture at \$3.1 per kilogram?
8. Two planes fly to the north and south from an airport. The plane, flying to the north, flies on 40 kilometers per hour faster than another one does. To the end of the third hour, the distance between them is 1800 km. Find the average speeds of planes.
9. The car, which speed is 96 km per hour, left the city 24 minutes after the truck and caught up it after 2 hours. Find the speed of the truck.
10. Asan and Ulan have 1532 soms. How many soms has Asan, if Ulan has 188 soms more?
11. Examine a system of equations to be solved for *x* and *y*, and in those cases when solutions exist, find them:

a) $\begin{cases} 3a - 2x - 5y = 8x + 11y + 2; \\ 13x - 2a + 7y = 7x + 3(2x + 3y + a); \end{cases}$

b) $\begin{cases} 2(5b + 3x) + 3(7 - 2y) = 6x + 14 - by; \\ 7x + 2y = 6x + 3y + 11b; \end{cases}$

c) $\begin{cases} 5(cx - 7) + 6y = cx + 2(7d + 3y); \\ x + 12d + 3y = 4(5y - 2cx + 7d). \end{cases}$

12. $\begin{cases} 4x + 5y = 3, \\ ax - 7y = 11. \end{cases}$ **13**. $\begin{cases} 2x + 3y = 10 - b, \\ 4x + by = 8. \end{cases}$

14. $\begin{cases} 3x + (8 - c)y = 7, \\ cx + 5y = 7. \end{cases}$ **15**. $\begin{cases} (10 - d)x + 3y = 2, \\ 8x + dy = 4. \end{cases}$

16. Solve system of linear equations and draw graph of equations:

a) $\begin{cases} 2x - 5y = 14; \\ 3x - 7y = 21; \end{cases}$ b) $\begin{cases} 3x + 2y = 16; \\ 7x + 3y = 21; \end{cases}$

c) $\begin{cases} 5x + 6y = 42; \\ x + 1.2y = 10.8; \end{cases}$ d) $\begin{cases} -3x + 8y = 36, \\ 6x - 16y = -72. \end{cases}$

17. At what values of the parameter *p* the following systems have a unique solution?

a) $\begin{cases} 20x + 25y = 2, \\ 4x + py = 3. \end{cases}$ b) $\begin{cases} 5x + 2y = 27, \\ 4x + py = 93. \end{cases}$

18. At what values of the parameter *p* the following systems doesn't have solutions?

a) $\begin{cases} 10x + 12.5y = 21, \\ 4x + py = 3. \end{cases}$ b) $\begin{cases} 11x - 2y = 27, \\ 4x + py = 9. \end{cases}$

19. At what values of the parameter *p* the following systems have infinitely many of solutions?

a) $\begin{cases} 20x + 25y = p, \\ 4x + 5y = 3. \end{cases}$ b) $\begin{cases} 21x - 12y = 27, \\ 14x + py = 18. \end{cases}$

20. Determine a number of system's solutions and draw a graph of equations:

a) $\begin{cases} 4x - 13y = 104; \\ 14x - 45.5y = 364. \end{cases}$ b) $\begin{cases} 6x - 19.5y = 156; \\ 22x - 71.5y = 574. \end{cases}$

c) $\begin{cases} 132x - 429y = 5\ 148; \\ 45x - 143y = 1\ 170. \end{cases}$

§7. Gauss method – the method of elimination

7.1. Introduction to the Gauss method

Suppose we know that the truck transported 2 tons of flour in 20 small and 25 large bags. To find the weight of each bag by means this information is impossible, as this problem has many solutions. For example, if the weight of a small bag 10 kg, while the weight of a large bag $(2000 - 20 \cdot 10):25 = 72$ kg; if the weight of a small bag 30 kg, the weight of a large bag $(2000 - 20 \cdot 30):25 = 56$ kg and so on.

This usually happens when the number of unknowns (the weights of a large bag and a small bag) is greater than the number of equations.
Therefore, in order to get an accurate answer, we need at an additional information. This can be information about the weight of an individual bag, other transportation etc.
Suppose that it is known, that the second truck transported 3 tons of flour in 38 small and 35 large bags. Using this information, we can create a system of equations to solve it, and write out the answer: the weight of a small bag 20 kg, a big — 64 kg.
If we have information on the third, fourth, etc. trucks, it could be used for verification. Let the third truck transported 2.5 tons of flour in 10 small and 40 large bags. Substituting the data on the weight of bags: $10 \cdot 20 + 40 \cdot 64 = 2\,760$, we get a mismatch. Consequently, in the information we have received, there is an error
Consider in more detail the solution to the problem of two trucks. Let x denote the weight of a small bag, and y the weight of a large bag, then we obtain a system of two equations:

$$\begin{cases} 20x + 25y = 2\,000, \\ 38x + 35y = 3\,000. \end{cases} \qquad (7.1)$$

We divide the first equation by 20 ($x + 1.25y = 100$) and multiplying by 38 (i.e., equating the coefficient of x in the first equation of (7.1) to 38), subtract from the second: $-12.5y = -800$. Hence $y = 64$. Substituting the value found in the first equation, we get $20x + 25 \cdot 64 = 2\,000$.
As result, $x = 20$.
Substitution of $x = 20$ and $y = 64$ in the second equation of system (7.1), we see that the solution found is true. The method that we are using is the method of Gauss. Note that

the system (7.1) can be written as the matrix equation $AX = B$, where $A=\begin{pmatrix} 20 & 25 \\ 38 & 35 \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \end{pmatrix}$, $B=\begin{pmatrix} 2000 \\ 3000 \end{pmatrix}$.

For large systems, solving by the method of Gauss, it is useful to write the augmented matrix of the system and describe elementary transformations of the equations of the system in a symbolic form. Consider, as an example, a solution of (7.1):

$$\left(\begin{array}{cc|c} 20 & 25 & 2000 \\ 38 & 35 & 3000 \end{array}\right) \xrightarrow[R_2 = r_2 - 38R_1]{R_1 = r_1 : 20} \left(\begin{array}{cc|c} 1 & 1.25 & 100 \\ 0 & -12.5 & -800 \end{array}\right) \xrightarrow[R_1 = r_1 - 1.25R_2]{R_2 = r_2 : (-12.5)} \left(\begin{array}{cc|c} 1 & 0 & 20 \\ 0 & 1 & 64 \end{array}\right)$$

Here the letter r (from the word row) represents a row of old matrix — the old row, R is the row of a new matrix — a new row. Record $R_1 = r_1{:}20$, for example, means that the new first row is obtained from the old first row by dividing by 20.

Exercises 7.1.

Solve a system and represent elementary transformations of the system of equations in symbolic form:

C. $\begin{cases} 2x - y = -21, \\ 3x + 5y = 14. \end{cases}$ **H**. $\begin{cases} 2x + 9y = 20, \\ 4x - 7y = -10. \end{cases}$

7.2. Gauss method

Process of solving systems by Gauss method consists of two parts: a direct way and reversed way. Each part is divided into steps.

Direct way. The first step is to choose the equation, put this equation on the first place, and using elementary transformations exclude the same unknown variable from all other equations.

In the next step the second and others equations, obtained in the first step of the system, are considered. We choose the equation and put it on the second place. Using elementary transformations, we exclude the same unknown variable from all other equations. And so on.

If by the transformation of the system the equations the equation $0{\cdot}x + 0{\cdot}y + ... + 0{\cdot}z = 0$ be obtained, this one may be deleted.

After finishing the direct way, the initial system equivalent to an obtained system, in which each subsequent equation has a smaller number of unknown variables. The three different situations are possible.

A. The obtained system has no solution — the last equation of the system has the form $0{\cdot}x = b$, where $b \neq 0$.

B. The obtained system has unique solution — the last equation has the form $ax = b$, where $a \neq 0$, and the number of equations of obtained system is equal to the number of unknown variables.

C. The obtained system has an infinitely many solutions. This situation occurs when number of unknowns in the last equation is more than one or, when any equation of the transformed system has less 2 (3, 4 or higher number) unknowns compared to previous, except the case **A.**

Reverse way. It is used in the cases **B** and **C** to write a solution of the system.

For the case **B**, on the first step we find the solution of the last equation of the system and put the found value to the previous equations. As a result, we obtain a system of $(n-1)$ equations, the last equation of which depends only on one variable. On the next steps we will continue the process until the values of other variables are found.

For the case **C** it is necessary to do the same way, only in those cases, when the last equation of the system has $k\ (k > 1)$ unknowns, it's necessary to choose $(k-1)$ unknowns any way, and express other unknown variables through them in the parametric form.

7.3. Problem

It is the result of the Elena's observations. First truck transported 2 345 kg flour in 23 small, 19 medium and 10 large bags. The second truck transported 2 835 kg of flour in 12 small, 25 medium and 16 large bags, the third truck transported 1 865 kg of flour in 34 small, 13 medium and 4 large bags. Find a weight of the each kind of the bag.

Solution

Let x be the weight of the small bag, y is the weight of the medium bag, z is the weight of the big bag. Then, we have a system of the linear algebraic equations:

$$\begin{cases} 23x + 19y + 10z = 2345, \\ 12x + 25y + 16z = 2835, \\ 34x + 13y + 4z = 1865. \end{cases}$$

First, it is handy rewrite the system in the form of an augmented matrix of coefficients

$$\left(\begin{array}{ccc|c} 23 & 19 & 10 & 2345 \\ 12 & 25 & 16 & 2835 \\ 34 & 13 & 4 & 1865 \end{array}\right) \begin{array}{c} R_1 = r_1 : 10 \\ \underline{R_2 = r_2 - 16R_1} \\ R_3 = r_3 - 4R_1 \end{array} \left(\begin{array}{ccc|c} 2.3 & 1.9 & 1 & 234.5 \\ -24.8 & -5.4 & 0 & -917 \\ 24.8 & 5.4 & 0 & 927 \end{array}\right)$$

(*We are used the coefficient 10 of 1st equation as simplest to exclude the coefficient of z in other equations*.)

$$\begin{array}{c} R_1 = r_1 \\ \underline{R_2 = r_2} \\ R_3 = r_3 + r_2 \end{array} \left(\begin{array}{ccc|c} 2.3 & 1.9 & 1 & 234.5 \\ -24.8 & -5.4 & 0 & -917 \\ 0 & 0 & 0 & 10 \end{array}\right).$$

We deal with the system that has no solution, or as it is also called with an inconsistent system. It means there some mistake in the Elena's observations.

Exercises 7.3.

C. According to Askar, on the first day, his store sold 8 kilograms of sweets, 5 kg of cookies and 7 kg of flour and earned 3 101 soms. On the second day, 2 kilograms of sweets, 7 kg of cookies and 9 kg of flour were sold; revenue amounted to 2 187 soms. On the third day, 14 kilograms of sweets, 3 kg of cookies and 5 kg of flour were sold; revenue amounted to 4 017 soms. Determine the prices of these products.

H. Solve the systems by the Gauss method:

$$\begin{cases} 2x + 8y + 6z = 20, \\ 5x + 2y - 2z = -2, \\ 3x - 6y - 8z = -21. \end{cases}$$

7.4. Problem

Gregory has stocks A, B, and C. He summed up the results of the financial year. Double stock return of B was $4000 more than the sum of 3 times A stock return and 3 times C stock return. Sum of 2 times A stock return and B stock return was $5 more than C stock return. The sum of stock return of A and double C stock return was $7000 less than 3 times B stock return.

Find the stock returns of A, B, and C.

Solution

Let a be A stock return, b is B stock return, c is C stock return (in thousand dollars). Then, a system of the linear algebraic equations have place: $\begin{cases} 2b - 4 = 3a + 3c, \\ 2a + b = c + 5, \\ a + 2c = 3b - 7. \end{cases}$

Rewrite the system in the standart form:

$$\begin{cases} 3a - 2b + 3c = -4, \\ 2a + b - c = 5, \\ a - 3b + 2c = -7. \end{cases}$$

Now, rewrite the system in the form of an augmented matrix of coefficients

$$\left(\begin{array}{ccc|c} 3 & -2 & 3 & -4 \\ 2 & 1 & -1 & 5 \\ 1 & -3 & 2 & -7 \end{array}\right) \xrightarrow[R_3 = r_3 + 3R_1]{\substack{R_1 = r_2 \\ R_2 = r_1 + 2R_1}} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 7 & 0 & 1 & 6 \\ 7 & 0 & -1 & 8 \end{array}\right) \xrightarrow[R_3 = r_3 - r_2]{\substack{R_1 = r_1 \\ R_2 = r_2}}$$

We are used the coefficient 1 of simplest second equation to exclude the coefficient of b in other equations.

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 7 & 0 & 1 & 6 \\ 0 & 0 & -2 & 2 \end{array}\right) \xrightarrow{R_3 = r_3 : (-2)} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 7 & 0 & 1 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

(*We start the reverse way*)

$$\begin{array}{c} R_1 = r_1 + r_3 \\ \underline{R_2 = r_2 - r_3} \\ R_3 = r_3 \end{array} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 4 \\ 7 & 0 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{R_2 = r_2 : 7} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\begin{array}{c} R_1 = r_1 - 2r_2 \\ \underline{R_2 = r_2} \\ R_3 = r_3 \end{array} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{c} R_1 = r_2 \\ \underline{R_2 = r_1} \\ R_3 = r_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Answer: $(1;\ 2;\ -1)$

Putting the found values in the initial system, we verify the correctness of found solution.

Exercises 7.4.

C. Solve the system by the Gauss method

$$\begin{cases} 2x + y + 2z = 3, \\ x - 2y - 3z = 1, \\ 3x + 2y + 4z = 5. \end{cases}$$

H. Nazar invested 200 000 soms in three deposit accounts at 9%, 10% and 10.5% interest rates, respectively, and a year later received 219 900 soms back. It is known that the first and second accounts together have 1.5 times more money than the third account. How much money was put into each account?

7.5. Problem

Aidai is planning to grow hens, ducks and turkeys. They will consume three types of food: A, B and C. According to the plan, a hen will consume one unit of each food weekly. A duck will consume one unit of A, two units of B and three units of

C. A turkey will consume 1, 3 and 5 units, correspondingly. Find the number of hens, ducks and turkeys, which Aidai can grow, knowing that 90 units A, 180 units B and 270 units C will be consumed weekly.

Solution

Let x be the number of hens, y is the number of ducks, z is the number of turkeys. Then, we have a system of the linear algebraic equations:
$$\begin{cases} x + y + z = 90, \\ x + 2y + 3z = 180, \\ x + 3y + 5z = 270. \end{cases}$$

First, to handy rewrite the system in the form of an augmented matrix of coefficients

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 90 \\ 1 & 2 & 3 & 180 \\ 1 & 3 & 5 & 270 \end{array}\right) \xrightarrow[R_3 = r_3 - R_1]{\substack{R_1 = r_1 \\ R_2 = r_2 - R_1}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 90 \\ 0 & 1 & 2 & 90 \\ 0 & 2 & 4 & 180 \end{array}\right) \Rightarrow$$

(*We remind, that the notation $R_1 = r_1$ means that first row remains first, $R_2 = r_2 - R_1$ means that for obtaining the new second row we substracted from the old second row of transformed matrix the first row and so on.*)

$$\xrightarrow[R_3 = r_3 - 2r_2]{\substack{R_1 = r_1 \\ R_2 = r_2}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 90 \\ 0 & 1 & 2 & 90 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus, the given system is equivalent to the following:
$$\begin{cases} x + y + z = 90, \\ \quad y + 2z = 90. \end{cases}$$

So, we have the case **C**. The last equation of the system has two unknowns, that's why one of them may be used as a parameter.

Let $z = a$. Then
$$\begin{cases} x + y = 90 - a, \\ \quad y = 90 - 2a, \end{cases}$$ we started the reverse way.

Put the value y:
$$\begin{cases} x + 90 - 2a = 90 - a, \\ \qquad y = 90 - 2a, \end{cases} \Leftrightarrow \begin{cases} x = a, \\ y = 90 - 2a. \end{cases}$$

Answer of the system: $x = a;\ \ y = 90 - 2a;\ \ z = a$ (a is arbitrarily). As x, y and z are numbers of hens, ducks and turkies, they should be non–negative and whole. Therefore, the number of hens and turkies a, number of ducks $90 - 2a$, where $a = 0; 1; \ldots ; 45.$
(The constraint $a \leq 45$ is from $0 \leq 90 - 2a$.)
Note. The reverse way can also be written in a matrix form.

Exercises 7.5.

Solve the systems by the Gauss method

C. $\begin{cases} 5x - 7y + 2z + t = 1, \\ 2y + 3z = 12, \\ 8z = 56. \end{cases}$ **H.** $\begin{cases} 2x + y - 2z = 3, \\ x - 2y - 3z = 1, \\ 4x - 3y - 8z = 5. \end{cases}$

Summary

1–6. Solve the systems by the Gauss method.

1. $\begin{cases} 2x - 3y + 4z = -4, \\ x + 5y - 2z = 15, \\ 4x + 2y + 6z = 10. \end{cases}$ 2. $\begin{cases} 3x + y + z = 4, \\ 2x + 2y + 3z = 1, \\ 3x - y - 2z = 7. \end{cases}$

3. $\begin{cases} x - 3y + 3z = -2, \\ 2x + y - 2z = 3, \\ 3x - y + z = 2. \end{cases}$ 4. $\begin{cases} 3x + 2y - z = 1, \\ x + y + 2z = 11, \\ 5x + 3y - 4z = 19. \end{cases}$

5. $\begin{cases} 3x - 2y + 4z = 9, \\ 2x + z = 3, \\ 5x - 2y + 5z = 13. \end{cases}$

6. The amount of $5000 was deposited in three investment companies with 6, 7 and 8 percent per annum. The total annual net income was $358, the income from the first two deposits exceeded the income from the third on $70. Find the amount of deposits to each company.

7. From a buyer who bought 2 bottles of milk, 3 bottles of kefir, and gave one empty bottle, the seller took $6. The purchases of a second buyer: 3 bottles of milk, 5 bottles of kefir, were estimated as $10.5; and of the third buyer, who bought 3 bottles of milk, 2 bottles of kefir, and who gave 2 empty bottles, were estimated as $5. What is the amount of the fourth buyer who bought 5 bottles of milk and gave 3 empty bottles?

Note The examples of solving the systems linear algebraic equations of fourth order by Gauss method you can see in Appendix.

Appendix. Gauss method

We consider the examples of solving the systems linear algebraic equations of big order in this appendix

$$1)\begin{cases} x_1 + 2x_2 - 3x_3 + 4x_4 = 7; \\ 2x_1 + 5x_2 + x_3 - 2x_4 = 5; \\ 3x_1 - 7x_2 + 4x_3 + 5x_4 = -11; \\ 7x_1 + 2x_2 - x_3 + 11x_4 = 6. \end{cases}$$

Solution

Let's rewrite the system in the form of an augmented matrix of coefficients

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 2 & 5 & 1 & -2 & 5 \\ 3 & -7 & 4 & 5 & -11 \\ 7 & 2 & -1 & 11 & 6 \end{array}\right) \begin{array}{l} R_1 = r_1 \\ R_2 = r_2 - 2R_1 \\ \hline R_3 = r_3 - 3R_1 \\ R_4 = r_4 - 7R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 1 & 7 & -10 & -9 \\ 0 & -13 & 13 & -7 & -32 \\ 0 & -12 & 20 & -1 & -43 \end{array}\right)$$

We remind, that the notation $R_1 = r_1$ means that first row remains first, $R_2 = r_2 - 2R_1$ means that for obtaining the second row we from the second row of transformed matrix substracted the doubled first row and so on.

$$\xrightarrow[\substack{R_3=r_3+13R_2\\ R_4=r_4+12R_2}]{R_2=r_2}\left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7\\ 0 & 1 & 7 & -10 & -9\\ 0 & 0 & 104 & -137 & -149\\ 0 & 0 & 104 & -137 & -151 \end{array}\right)$$

$$\xrightarrow[R_4=r_4-R_3]{R_3=r_3}\left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7\\ 0 & 1 & 7 & -10 & -9\\ 0 & 0 & 104 & -137 & -149\\ 0 & 0 & 0 & 0 & -2 \end{array}\right)$$

Thus, the given system is equivalent to the following:

$$\begin{cases} x_1+2x_2-3x_3+4x_4=7;\\ x_2+7x_3-10x_4=-9;\\ 104x_3-137x_4=-149;\\ 0=-2. \end{cases}$$

I.e. we deal with the system that has no solution, or as it is also called with an inconsistent system.

2) $\begin{cases} x_1-x_2+2x_3-3x_4=1;\\ x_1+4x_2-x_3-2x_4=-2;\\ x_1-4x_2+3x_3-2x_4=-2;\\ x_1-8x_2+5x_3-2x_4=-2. \end{cases}$

Solution

We use the direct way of the Gauss method.

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 1\\ 1 & 4 & -1 & -2 & -2\\ 1 & -4 & 3 & -2 & -2\\ 1 & -8 & 5 & -2 & -2 \end{array}\right)\xrightarrow[\substack{R_3=r_3-R_1\\ R_4=r_4-R_1}]{\substack{R_1=r_1\\ R_2=r_2-R_1}}\left(\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 1\\ 0 & 5 & -3 & 1 & -3\\ 0 & -3 & 1 & 1 & -3\\ 0 & -7 & 3 & 1 & -3 \end{array}\right)$$

$$\xrightarrow[\substack{R_3=r_3-R_2\\ R_4=r_4-R_2}]{R_2=r_2}\left(\begin{array}{cccc|c}1 & -1 & 2 & -3 & 1\\ 0 & 5 & -3 & 1 & -3\\ 0 & -8 & 4 & 0 & 0\\ 0 & -12 & 6 & 0 & 0\end{array}\right)\xrightarrow[R_4=r_4-6R_3]{R_3=r_3:4}$$

$$\left(\begin{array}{cccc|c}1 & -1 & 2 & -3 & 1\\ 0 & 5 & -3 & 1 & -3\\ 0 & -2 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right)$$

So, we have the case with infinite number of solutions.
The last equation of the system

$$\begin{cases} x_1 - x_2 + 2x_3 - 3x_4 = 1;\\ 5x_2 - 3x_3 + x_4 = -3;\\ -2x_2 + x_3 = 0\end{cases}$$

has two unknowns, that's why one of them may be used as a parameter.

Let $x_2 = a$. Then $\begin{cases} x_1 - x_2 + 2x_3 - 3x_4 = 1;\\ -3x_3 + x_4 = -3 - 5a;\\ x_3 = 2a\end{cases}$ we started the reverse way.

Put the value x_3 :

$$\begin{cases} x_1 - 3x_4 = 1 + a - 4a;\\ x_4 = -3 - 5a + 6a;\end{cases} \Leftrightarrow \begin{cases} x_1 = -8;\\ x_4 = -3 + a.\end{cases}$$

Answer: $x_1 = -8; x_2 = a; x_3 = 2a; x_4 = -3+a$ (a is any number).

Note. The reverse way can also be written in a matrix form.

3) $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 7;\\ 2x_1 + x_2 + 2x_3 + 3x_4 = 6;\\ 3x_1 + 2x_2 + x_3 + 2x_4 = 7;\\ 4x_1 + 3x_2 + 2x_3 + x_4 = 18.\end{cases}$

Solution

$$\left(\begin{array}{cccc|c}1 & 2 & 3 & 4 & 7\\ 2 & 1 & 2 & 3 & 6\\ 3 & 2 & 1 & 2 & 7\\ 4 & 3 & 2 & 1 & 18\end{array}\right)\xrightarrow[\substack{R_3=r_3-3R_1\\ R_4=r_4-4R_1}]{\substack{R_1=r_1\\ R_2=r_2-2R_1}}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 7 \\ 0 & -3 & -4 & -5 & -8 \\ 0 & -4 & -8 & -10 & -14 \\ 0 & -5 & -10 & -15 & -10 \end{array}\right) \begin{array}{l} R_2 = r_4 : (-5) \\ \underline{R_3 = r_3 + 4R_2} \\ R_4 = r_4 + 3R_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 7 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 2 & -6 \\ 0 & 0 & 2 & 4 & -2 \end{array}\right) \begin{array}{l} \underline{R_3 = r_3 : 2} \\ R_4 = r_4 : 2 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 7 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

(We start the reverse way)

$$\begin{array}{l} R_1 = r_1 - 4r_4 \\ \underline{R_2 = r_2 - 4r_4} \\ R_3 = r_3 - 2r_4 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 19 \\ 0 & 1 & 2 & 0 & 11 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right) \quad \begin{array}{l} R_1 = r_1 - 3r_3 \\ \underline{R_2 = r_2 - 2r_2} \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right) \underline{R_1 = r_1 - 2r_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

Answer: $(2; 1; 5; -3)$

Putting the found values in the original system, we verify the correctness of found solution.

Exercises

1–4. Solve the systems by the Gauss method.

1. $\begin{cases} x_1 + 2x_2 - 3x_3 + 5x_4 = 1; \\ x_1 + 3x_2 - 13x_3 + 22x_4 = -1; \\ 3x_1 + 5x_2 + x_3 - 2x_4 = 5; \\ 2x_1 + 3x_2 + 4x_3 - 7x_4 = 4. \end{cases}$

2. $\begin{cases} x_1 - 2x_2 + 3x_3 - 4x_4 = 2; \\ 3x_1 + 3x_2 - 5x_3 + x_4 = -3; \\ -2x_1 + x_2 + 2x_3 - 3x_4 = 5; \\ 3x_1 + 3x_3 - 10x_4 = 8. \end{cases}$

3. $\begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 = 1; \\ 2x_1 - x_2 - 2x_3 - 3x_4 = 2; \\ 3x_1 + 2x_2 - x_3 + 2x_4 = 21; \\ 2x_1 - 3x_2 + 2x_3 + x_4 = 11. \end{cases}$

4. $\begin{cases} x_1 - x_2 + 2x_3 + 2x_4 + 7x_5 = 1; \\ 2x_1 - 3x_2 + 2x_3 + x_4 - 2x_5 = -2; \\ 3x_1 - 5x_2 + 2x_3 - 3x_4 + 8x_5 = 7; \\ -4x_1 + 12x_2 + 8x_3 + 10x_4 + 11x_5 = 52. \end{cases}$

Summary for chapters 1–7

1. KSL is a triangle. M is the midpoint of KL. $\overline{KS} = \overline{a}$, $\overline{KL} = \overline{b}$.

a) Express $\overline{KM}$ in the terms of $\overline{a}$ and $\overline{b}$.

b) Express $\overline{SM}$ in the terms of $\overline{a}$ and $\overline{b}$

Give your answers in its simplest forms.

2. The points $A(6, 8)$, $B(-1,2)$, $C(11,-2)$ are given. The point K that is divide the segment AC in the ratio 1:4 and the point N that is divide the segment BC in the ratio 1:3.

a) Find an area of a triangle KNC.

b) Find a coordinate of the point T that is the vertex of the parallelogram $KNCT$.

c) Find a decomposition of the vector $\overline{NT}$ in basis of vectors $\overline{AC}$ and $\overline{BC}$.

d) Find an equation of the straight line AL that is parallel to BC.

e) Find an equation of the straight line BD that is perpendicular to the straight line AC.

3. A Web server has an original value of $10 000 and is to be depreciated linearly over 5 years with a $3 000 scrap value.

a) Find an expression giving the book value at the end of year t.

b) What will be the book value of the server at the end of the second year?

4. The quantity demanded of a certain model of DVD player is 8 000 units when the unit price is \$260. At a unit price of \$200, the quantity demanded increases to 10 000 units. The manufacturer will not market any players if the price is \$100 or lower. However, for each \$50 increase in the unit price above \$100, the manufacturer will market an additional 1000 units. Both the demand and the supply equations are known to be linear.

a) Find the demand equation.

b) Find the supply equation.

c) Find the equilibrium quantity and price.

5. The costs of preparation for the release of a new type of product is \$2 044. Plot the graph and determine the profit zone, if it is known that the average variable cost for the first 400 units is \$4 and for subsequent is \$1.7, while a price (in \$) equals

a) *14.22;* b) *5.7;* c) $36 - 0.04q$

6. Almaz and Denis went shopping for Halloween treats. Almaz bought 3 chocolate pumpkins and 8 candy witches. He spent 705 soms. Denis bought 4 chocolate pumpkins and 6 candy witches. He spent 660 soms. Find unit price of each item purchased.

7. Investigate the system on solvability and write solutions for each particular case.

$$a)\begin{cases} 3x - 2y = 5, \\ 6x - 4y = a; \end{cases} \qquad b)\begin{cases} 2x + (m-1)y = 3, \\ (m+1)x + 4y = -3. \end{cases}$$

§8. Determinants. Cramer's rule

8.1. Determinants

Definition. *Determinant of the square matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \qquad (8.1)$$

is a number that is denoted as $\det A$ *or*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix},$$

and is computed according the rule:

a) determinant of the 1st order matrix is equal to the element of matrix: $\det(a) = a$;

b) determinant of matrix of n-order is equal to:

$\det A = a_{11}\det A_1 - a_{21}\det A_2 + a_{31}\det A_3 - \dots + (-1)^{n-1}a_{n1}\det A_n$

where $\det A_k$ *is a determinant of the matrix of order* $(n-1)$*, which is obtained from matrix* A *by deleting the k-th row and first column.*

It may be argued that there is not everything right. To calculate the determinant of matrix of order *n*, it is necessary to use the determinants of matrices of order $(n-1)$, which are unknown. According to the definition, the determinant of the matrix of order $(n-1)$ will be known; if there are known determinants of matrices of order $(n-2)$ and so on. Fortunately, this chain has the end — the determinant of the first order matrix is known from the point *a).*

The method used in the definition is called the decomposition of the determinant by the first column, the numbers $detA_k$ are called minors, corresponding to the elements a_{k1}.

Problem *Calculate the determinant* $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution

From the definition it follows that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot det(d) - c \cdot det(b) = a \cdot d - c \cdot b.$$

We showed that the determinant of the 2-order is equal to multiplication of matrix elements standing on the main diagonal minus the product of elements of the second diagonal.

Consequently, it is not necessary to use the definition each time, calculating the determinants of the second order.

Exercises 8.1. Calculate the determinant:

C. $\begin{vmatrix} 8 & -12 \\ 7 & 3 \end{vmatrix}$ **H.** $\begin{vmatrix} -4 & 21 \\ 3 & 15 \end{vmatrix}$

8.2. Problem

Calculate the determinant $\begin{vmatrix} 3 & -2 & -10 \\ 7 & 8 & -2 \\ -10 & 0 & 9 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 3 & -2 & -10 \\ 7 & 8 & -2 \\ -10 & 0 & 9 \end{vmatrix} = 3\begin{vmatrix} 8 & -2 \\ 0 & 9 \end{vmatrix} - 7\begin{vmatrix} -2 & -10 \\ 0 & 9 \end{vmatrix} +$$

$$+(-10)\begin{vmatrix} -2 & -10 \\ 8 & -2 \end{vmatrix} =$$

$$=3(72-0)-7(-18-0)+(-10)(4-(-80)) =$$

$$=216+126-840 = -498.$$

Exercises 8.2. Calculate the determinant:

C. $\begin{vmatrix} 5 & -12 & 10 \\ -4 & 8 & 2 \\ 6 & 4 & 3 \end{vmatrix}$ **H.** $\begin{vmatrix} 15 & 2 & -20 \\ 4 & -8 & 12 \\ 6 & -4 & 13 \end{vmatrix}$

8.3. Cramer's rule

If the determinant of the coefficient matrix of the system of linear algebraic equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n. \end{cases} \quad (8.2)$$

is different from zero, the system has an unique solution $\{\Delta(1)/\Delta;\ \Delta(2)/\Delta;\ \ldots;\ \Delta(n)/\Delta\}$, where Δ is the determinant of the coefficients of system (2), $\Delta(k)$ is the determinant, which is obtained if in the determinant Δ column k is replaced by a column of free terms of system (2), that is a column of numbers b_m.

The proof of the theorem you can see in Appendix.

Problem

A pastry shop sells cakes. On Thursday, when it sold 12 small, 6 medium and one large cake, revenue was \$330. Corresponding data on Friday was 15 small; 9 medium; 3 large; \$495. On Saturday: 20 small; 13 medium; 5 large; \$710. Determine the price of each kind of the cake.

Solution

Let denote the price of the small cake as *x*, medium as *y*, large as *z*.

We calculate the determinant of the coefficient matrix of the system

$$\begin{cases} 12x + 6y + z = 330, \\ 15x + 9y + 3z = 495, \\ 20x + 13y + 5z = 710, \end{cases} \qquad (8.3)$$

using an expansion by the first column:

$$\Delta = \begin{vmatrix} 12 & 6 & 1 \\ 15 & 9 & 3 \\ 20 & 13 & 5 \end{vmatrix} = 12\begin{vmatrix} 9 & 3 \\ 13 & 5 \end{vmatrix} - 15\begin{vmatrix} 6 & 1 \\ 13 & 5 \end{vmatrix} + 20\begin{vmatrix} 6 & 1 \\ 9 & 3 \end{vmatrix} =$$

$$= 12(9{\cdot}5 - 13{\cdot}3) - 15(6{\cdot}5 - 13{\cdot}1) + 20(6{\cdot}3 - 9{\cdot}1) =$$

$$= 12{\cdot}6 - 15{\cdot}17 + 20{\cdot}9 = -3.$$

Replacing the first column of the coefficient matrix by the right-hand side, we obtain $\Delta_x = \begin{vmatrix} 330 & 6 & 1 \\ 495 & 9 & 3 \\ 710 & 13 & 5 \end{vmatrix} =$

$$= 330\begin{vmatrix} 9 & 3 \\ 13 & 5 \end{vmatrix} - 495\begin{vmatrix} 6 & 1 \\ 13 & 5 \end{vmatrix} + 710\begin{vmatrix} 6 & 1 \\ 9 & 3 \end{vmatrix}.$$

Notice, that all the determinants of the second order are calculated with the value of Δ, we find that $\Delta_x = 330{\cdot}6 - 495{\cdot}17 + 710{\cdot}9 = -45$.

Therefore, by Cramer's rule $x = \Delta_x/\Delta = -45/(-3) = 15$.

Replacing the second column of the coefficient matrix by the right-hand side, we obtain $\Delta_y = \begin{vmatrix} 12 & 330 & 1 \\ 15 & 495 & 3 \\ 20 & 710 & 5 \end{vmatrix} =$

$$= 12\begin{vmatrix} 495 & 3 \\ 710 & 5 \end{vmatrix} - 15\begin{vmatrix} 330 & 1 \\ 710 & 5 \end{vmatrix} + 20\begin{vmatrix} 330 & 1 \\ 495 & 3 \end{vmatrix} =$$

$$= 12(495{\cdot}5 - 710{\cdot}3) - 15(330{\cdot}5 - 710{\cdot}1) + 20(330{\cdot}3 - 495{\cdot}1)$$

$$= 12{\cdot}345 - 15{\cdot}940 + 20{\cdot}495 = -60.$$

Then, $y = (-60)/(-3) = 20$.

At the same way: $\Delta_z = \begin{vmatrix} 12 & 6 & 330 \\ 15 & 9 & 495 \\ 20 & 13 & 710 \end{vmatrix} =$

$$= 12\begin{vmatrix} 9 & 495 \\ 13 & 710 \end{vmatrix} - 15\begin{vmatrix} 6 & 330 \\ 13 & 710 \end{vmatrix} + 20\begin{vmatrix} 6 & 330 \\ 9 & 495 \end{vmatrix} =$$

$=12(9\cdot 710 - 13\cdot 495) - 15(6\cdot 710 - 13\cdot 330)+$
$+ 20(6\cdot 495 - 9\cdot 330) = 12\cdot(-45) - 15\cdot(-30) + 20\cdot 0 = -90,$
and $z = (-90)/(-3) = 30.$

So it turned out that a small cake costs \$15, medium cake costs \$20, and large cake costs \$30.

Exercises 8.3. Solve system:

C. $\begin{cases} 2x - y + 7z = -21, \\ 3x + 5y + 2z = 33, \\ 7x - 2y + 3z = 0. \end{cases}$ **H.** $\begin{cases} 2x + 8y + 6z = 20, \\ 4x + 2y - 2z = -2, \\ 3x - y + z = 11. \end{cases}$

8.4. Area of a triangle

Many geometric problems are easier when to solve them using coordinates. One of the most striking examples of this kind is the problem of determining the area of a triangle.

Let there be a triangle with vertices at the points $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$.

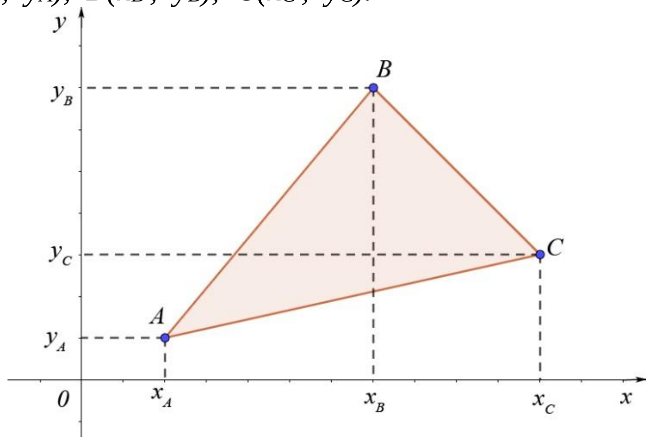


Figure 8.1

The area of the triangle ABC is the difference between the sum of the areas of the trapezoids y_BBCy_C and y_AACy_C and the area of the trapezoid y_BBAy_A .

Wherein,

area of the trapezoid $y_B B C y_C$ equals $\frac{x_B + x_C}{2}(y_B - y_C)$;

area of the trapezoid $y_A A C y_C$ equals $\frac{x_C + x_A}{2}(y_C - y_A)$.

area of the trapezoid $y_B B A y_A$ equals $\frac{x_A + x_B}{2}(y_B - y_A)$;

Hence,

$$S_{ABC} = \frac{x_B + x_C}{2}(y_B - y_C) + \frac{x_C + x_A}{2}(y_C - y_A) - \frac{x_A + x_B}{2}(y_B - y_A)$$

$$= \frac{1}{2}(x_B y_B - x_B y_C + x_C y_B - x_C y_C) +$$

$$+ \frac{1}{2}(x_C y_C - x_C y_A + x_A y_C - x_A y_A) -$$

$$- \frac{1}{2}(x_A y_B - x_A y_A + x_B y_B - x_B y_A) =$$

$$= -\frac{1}{2}(x_B y_C - x_C y_B) + \frac{1}{2}(x_A y_C - x_C y_A) - \frac{1}{2}(x_A y_B - x_B y_A).$$

Please note, that the expressions in the last three brackets can be written in the form of second-order determinants, which can then be assembled into a third-order determinant:

$$S_{ABC} = -\frac{1}{2}\left(1 \cdot \begin{vmatrix} x_B & y_B \\ x_C & y_C \end{vmatrix} - 1 \cdot \begin{vmatrix} x_A & y_A \\ x_C & y_C \end{vmatrix} + 1 \cdot \begin{vmatrix} x_A & y_A \\ x_B & y_B \end{vmatrix}\right)$$

$$S_{ABC} = -\frac{1}{2} \cdot \begin{vmatrix} 1 & x_A & y_A \\ 1 & x_B & y_B \\ 1 & x_C & y_C \end{vmatrix}$$

Note that for another arrangement of the vertices of the triangle, the sign of the determinant may change. But this is not a cause for concern: you need to take half the absolute value (modulus) of the corresponding determinant.

So, it is proved that the area of a triangle with vertices at points $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$ can be calculated by the formula:

$$S = |0.5\det T|, \text{ where } \det T = \begin{vmatrix} 1 & x_A & y_A \\ 1 & x_B & y_B \\ 1 & x_C & y_C \end{vmatrix}.$$

Exercises 8.4.

Calculate the area of a triangle whose vertices have coordinates:

C. *(5, – 12), (10, 2), (8, – 4);* **H.** *(6, 4), (3, 15), (2, 20).*

8.5. Eigenvalues and Eigenvectors

Consider the matrix equation

$$AX = kX, \qquad (8.4)$$

where A is a square matrix, k is any number.

Zero-column matrix is the solution of the equation (8.4). Nonzero solutions are interesting in this situation.

Definition

Number k is called an eigenvalue of the square matrix A, if there is a nonzero column-matrix X such that $AX = kX$. If k is the eigenvalue of the matrix A, then every non-zero solution of equation (8.4) is called its eigenvector of the matrix A, corresponding to the eigenvalue k.

Now, we will search the eigenvalues.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$.

Than equation (8.4) is equivalent to the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = kx_1; \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = kx_2; \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = kx_n; \end{cases} \Leftrightarrow$$

$$\begin{cases} (a_{11} - k)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0; \\ a_{21}x_1 + (a_{22} - k)x_2 + \ldots + a_{2n}x_n = 0; \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn} - k)x_n = 0. \end{cases} \qquad (8.5)$$

Homogeneous system (a system with zero right part) always has zero solution. Therefore, the system (8.5) has a solution other than zero solutions if and only if its coefficients' matrix has a determinant equal to zero.
So, the number k is the eigenvalue if it equals to zero determinant

$$\begin{vmatrix} a_{11}-k & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-k & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-k \end{vmatrix} = C(k).$$

Function $C(k)$ is called characteristic polinom of the matrix A. As degree of polinom $C(k)$ equals n, the equation $C(k) = 0$ has n roots.
Now to find the eigenvectors of the matrix corresponding to the eigenvalue k, we should insert value of k into (8.5) and solve the system.

Example

To find eigenvalues and eigenvectors of a matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$, construct the characteristic equation $\begin{vmatrix} 2-k & 1 \\ 2 & 3-k \end{vmatrix} = 0.$
It equivalents to quadratic equation

$$(2-k)(3-k) - 2 = 0 \;\Leftrightarrow\; k^2 - 5k + 4 = 0.$$

Roots of the equation are $k_1 = 1;\ k_2 = 4$.
Substituting $k = 1$ into the system of the type (8.5) we find the corresponding eigenvectors:

$$\begin{cases} (2-1)x + y = 0; \\ 2x + (3-1)y = 0; \end{cases} \Leftrightarrow \begin{cases} x + y = 0; \\ 2x + 2y = 0; \end{cases} \Leftrightarrow \begin{cases} x + y = 0; \\ 0 = 0. \end{cases}$$

If $x = a$, then $y = -a$. Therefore, eigenvectors corresponding to eigenvalue $k = 1$, these are the vectors $\begin{pmatrix} a \\ -a \end{pmatrix}$, where a is arbitrary number.

If $k = 4$, then

$$\begin{cases} (2-4)x + y = 0; \\ 2x + (3-4)y = 0; \end{cases} \Leftrightarrow \begin{cases} -2x + y = 0; \\ 2x - y = 0; \end{cases} \Leftrightarrow \begin{cases} y = 2x; \\ 0 = 0. \end{cases}$$

It means, the eigenvectors correspond to eigenvalue $k = 4$ are vectors $\begin{pmatrix} b \\ 2b \end{pmatrix}$, where b is arbitrary number.

Exercises 8.5.

Find eigenvalues and eigenvectors of a matrix:

C. $\begin{pmatrix} 0 & 3 & 6 \\ -2 & 1 & 12 \\ 0 & 1 & 2 \end{pmatrix}$ **H.** $\begin{pmatrix} 7 & -1 \\ 8 & 1 \end{pmatrix}$

Summary

1–10.Calculate the determinants:

1. $\begin{vmatrix} -3 & 2 \\ 4 & 8 \end{vmatrix}$ 2. $\begin{vmatrix} -7 & -15 \\ 1 & 3 \end{vmatrix}$ 3. $\begin{vmatrix} 2-x & 1 \\ 3 & x \end{vmatrix}$

4. $\begin{vmatrix} 8389 & 16778 \\ 234314 & 468628 \end{vmatrix}$ 5. $\begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 5 \\ 5 & 3 & 4 \end{vmatrix}$ 6. $\begin{vmatrix} 1 & -3 & 5 \\ 7 & 2 & 3 \\ 5 & -1 & -2 \end{vmatrix}$

7. $\begin{vmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{vmatrix}$ 8. $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{vmatrix}$ 9. $\begin{vmatrix} 2 & 1 & 0 & 3 \\ -1 & 2 & 0 & 4 \\ -2 & 3 & 0 & 5 \\ -3 & 4 & 1 & 6 \end{vmatrix}$

10. $\begin{vmatrix} 1 & 0 & 0 & 2 & 3 \\ 3 & 0 & 0 & 4 & 1 \\ 0 & 5 & 6 & 0 & 1 \\ 0 & 7 & 8 & 0 & -2 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix}$

11–14. Solve systems:

11. $\begin{cases} 2x - 3y + 7z = 28, \\ 3x + 5y - 10z = -29, \\ 7x - 3z = 5. \end{cases}$ 12. $\begin{cases} 3x + y + z = 3, \\ 2x + 3y + 4z = 3, \\ 4x + 9y + 16z = 11. \end{cases}$

13. $\begin{cases} 10x - 3y - 7z = 16, \\ 3x + 2y - 15z = -5, \\ 7y - 5z = 6. \end{cases}$ 14. $\begin{cases} x + y + 6z = 3, \\ -x + 3y + z = 3, \\ 4x + 8y + 30z = 21. \end{cases}$

15. The Alfa Company manufactures three types of lamps, labeled A, B, and C. Each lamps processed in two departments, I and II. Time requirement and profit per unit for each lamp type is as follow:

	A	B	C
Man-hours in I	2	3	1
Man-hours in II	4	2	3
Profit	\$5	\$4	\$3

How many lamps of each type were produced, if company spent 615 man-hours in I, 1040 man-hours in II and received \$1330 profit?

16. The Beta Company manufactures products K, L, and M. Each product processed in three departments with time requirements per unit as follow:

	K	L	M
Assembling	2	3	2
Painting	3	1	3
Finishing	2	1.5	1

How many product of each type were manufactured, if company spent 1832 man-hours for assembling, 1558 man-hours for painting and 1136 man-hours for finishing?

17. Calculate the area of a triangle whose vertices have coordinates:

1) $(-5, 2),\ (8, 12),\ (5, 14)$.   2) $(6, -4), (-3, 5),\ (2, 2)$.

18 – 19. Find eigenvalues and eigenvectors of a matrix:

18. $\begin{pmatrix} 3 & 5 \\ 6 & 4 \end{pmatrix}$,   **19.** $\begin{pmatrix} 1 & 3 & 0 \\ 7 & 1 & 5 \\ 0 & -4 & 1 \end{pmatrix}$.

Appendix. Cramer's rule

If the determinant of the coefficient matrix of the system of linear algebraic equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n. \end{cases} \qquad (8.6)$$

is different from zero , the system has a solution $\{\Delta(1)/\Delta;\quad \Delta(2)/\Delta;\ \ldots\ ;\Delta(n)/\Delta\}$,
where Δ is the determinant of the coefficients of system (8.6), $\Delta(k)$ is the determinant, which is obtained if in the determinant Δ column k is replaced by a column of free terms of system (8.6), that is a column of numbers b_m.

Proof

We provide a proof for the system of the second order. Note that the proposed proof can be almost literally used to prove the theorem in the case of a system of an arbitrary order.

From the coefficients and the right-hand side of the system of linear algebraic equations

$$\begin{cases} ax + by = e, \\ cx + dy = f, \end{cases} \qquad (8.7)$$

we create three determinants:

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix};\ \Delta(1) = \begin{vmatrix} e & b \\ f & d \end{vmatrix};\ \Delta(2) = \begin{vmatrix} a & e \\ c & f \end{vmatrix}.$$

At the first stage of the proof we create the determinant from the coefficients of system (8.7):

$$K_1 = \begin{vmatrix} a & b & e \\ a & b & e \\ c & d & f \end{vmatrix}.$$

In the first and second rows of the determinant are the coefficients of the first equation of system (8.7), in the third row – of the second equation.

As we know, determinants with same rows are equal to zero. We write this fact, previously expanding determinant K_1 by the first row :

$$a\begin{vmatrix} b & e \\ d & f \end{vmatrix} - b\begin{vmatrix} a & e \\ c & f \end{vmatrix} + e\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0. \qquad (8.8)$$

Interchanging columns in the first determinant of equality (8.8) and using the notations of these determinants, we rewrite equality (8.8) as $-a\Delta(1) - b\Delta(2) + e\Delta = 0.$

We transfer the last term behind the equal sign , divide the resulting expression by $-\Delta$: $a\Delta(1)/\Delta + b\Delta(2)/\Delta = e$, and see that numbers $\Delta(1)/\Delta$ and $\Delta(2)/\Delta$ are the roots of the first equation of system (8.7).

At the second stage of the proof, we create the determinant

$K_2 = \begin{vmatrix} c & d & f \\ a & b & e \\ c & d & f \end{vmatrix}$. It is easy to notice that unlike K_1, in the first row of this determinant are the coefficients of the second equation of the system.

Expanding K_2 by the first row , and doing the same as in the first stage of transformation , we see that numbers $\Delta(1)/\Delta$ and $\Delta(2)/\Delta$ are the roots of the 2nd equation of system (8.7) too. The theorem is proved.

§9. Cramer - Gauss Method for Solving Systems of Linear Algebraic Equations

We all know that Gauss elimination method and Cramer's rule are usually used for solving systems of linear algebraic equations. However, there is now a popular opinion that the Cramer's rule became obsolete. For example, this method is not even described in many American Advanced Math textbooks. We can understand the supporters of this approach. Cramer's rule may be used only if the determinant of the system is not equal to zero. Moreover, to solve a system of N equations with N unknowns it is necessary to compute $N + 1$ determinants of the order N, which is a difficult and rather tedious task, when N is equal to three or more. Therefore, they propose to use Cramer's rule only in theoretical studies.

In our opinion, we should not abandon Cramer's rule. Some modifications of this method allow maintaining its values, while getting rid of its inherent flaws.

While solving systems by Cramer's rule corresponding determinants are computed independently. However, we can use the fact that the same minor determinants appear in all of these determinants. In order to do this, the determinant of the coefficients of the system must be calculated using the decomposition by the first column.

Also, to find the unknowns of the system we should use each value found to reduce the order of the system and to find the next value. This approach is typical for the Gauss elimination method. As a result, it will be sufficient to calculate only the determinant of the coefficient matrix in order to solve the system.

We called the method for solving systems based on these principles the Cramer-Gauss method.

The process of solving systems of linear algebraic equations by Cramer - Gauss method is quite transparent and simple. In this regard, we believe that it can be included in a math course for secondary school.

9.1. Problem

Karim has 60 bills in five-dollar, ten-dollar and twenty-dollar denominations. The number of twenties is four times the number of fives, while the total value of the money is $845. Find the number of each type of bill.

Solution

Let x be a number bills in five-dollar denomination, y is ten-dollar denomination, z is twenty-dollar denomination. Then, we have a system of the linear algebraic equations:

$$\begin{cases} x + y + z = 60, \\ z = 4x, \\ 5x + 10y + 20z = 845, \end{cases} \Rightarrow \begin{cases} x + y + z = 60, \\ 4x - z = 0, \\ 5x + 10y + 20z = 845. \end{cases} \qquad (9.1)$$

Using a decomposition by the first column we calculate the determinant of system coefficients:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 0 & -1 \\ 5 & 10 & 20 \end{vmatrix} = 1\begin{vmatrix} 0 & -1 \\ 10 & 20 \end{vmatrix} - 4\begin{vmatrix} 1 & 1 \\ 10 & 20 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} =$$

$$= 1[0 \cdot 20 - 10 \cdot (-1)] - 4[1 \cdot 20 - 10 \cdot 1] + 5[1 \cdot (-1) - 0 \cdot 1] =$$

$$= 1 \cdot 10 - 4 \cdot 10 + 5 \cdot (-1) = -35.$$

Replacing the first column of the coefficient matrix by the right-hand side, we obtain

$$\Delta_x = \begin{vmatrix} 60 & 1 & 1 \\ 0 & 0 & -1 \\ 845 & 10 & 20 \end{vmatrix} =$$

$$= 60\begin{vmatrix} 0 & -1 \\ 10 & 20 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 10 & 20 \end{vmatrix} + 845\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}$$

Noticing that all the determinants of the second order were calculated to find the value of Δ, we use them and find that $\Delta_x = 60 \cdot 10 - 0 \cdot 10 + 845 \cdot (-1) = -245.$

Therefore, by Cramer's rule $x = \Delta_x / \Delta = -245/(-35) = 7$.

Then, it is possible to continue the solving by the Cramer's rule and to find Δ_y, Δ_z, … .

However, we can see an easier way. Using the meaning $x = 7$ by second equation of the system (9.1), we find value of z: $z = 4x = 28$. Then, from the first equation $x + y + z = 60$, we obtain $y = 60 - 7 - 28 = 25$. So, we determine that Karim has 7 bills in five-dollar denomination, *25* — in ten-dollar denomination, *28* — in twenty-dollar denomination.

To solve this problem we used Cramer-Gauss method.

Exercises 9.1. Solve systems:

C. $\begin{cases} 17x - 11y = 161, \\ 7x + 13y - 9z = 54, \\ 22x - 19y + 5z = 250. \end{cases}$ **H.** $\begin{cases} 7x - 11y + 8z = -12, \\ 9x + 10y - 13z = 220, \\ 8x + 13y = 127. \end{cases}$

9.2. Problem

Solve the system $\begin{cases} 521x + 327y = 4301, \\ 267x + 741y = 3351. \end{cases}$ (9.2)

Of course, you can start to solve it right away, but it will be more interesting and useful for students to get this system as a problem for which the solution to the system (9.2) is used: *The shop trades in chocolate sweets and candy. Per 1st day, when 521 kg of chocolate sweets and 327 kg of candy were sold, the proceeds have made \$ 4301. Per 2nd day, when 267 kg of chocolate sweets were sold and 741 kg of candy are purchased, the proceeds have made \$ 3351. Determine the prices.*

Solution

Determinant of system (9.2)

$$\Delta = \begin{vmatrix} 521 & 327 \\ 267 & 741 \end{vmatrix} = 521 \cdot 741 - 267 \cdot 327 = -441770.$$

To find x, coefficients of x we replace on the free terms and calculate Δ_x: $\Delta_x = 4301 \cdot 741 - 3351 \cdot 327 = -3092390.$

Then, under the Cramer's rule

$x = \Delta_x / \Delta = -3092390/(-441770) = 7.$

Further, following the Cramer-Gauss method, we exclude one of the equations of system (9.2) (more complicated is recommended, in this particular case the equations are equivalent – then any. Let it be the 2nd.). Substituting the value found in the 1st step $x = 7$ to the 1st equation we obtain

$521 \cdot 7 + 327y = 4301.$

Then $327y = 4301 - 3647 = 654$ and $y = 2.$

Exercises 9.2.

C. Saadat sells sweets and cookies. On the first day, she sold 17 kilograms of sweets and 22.5 kg of cookies. On the second day, she sold 25.2 kg of sweets and 19 kg of cookies. Determine the prices of sweets and cookies, knowing that the revenue on the first day was equal to 9210 soms, on the second day — equal to 10868 soms.

H. Diana put 80 000 soms into two accounts and a year later received 88 190 soms back. The interest rate on the first account is 9%, on the second 12%. How much money was put into each account?

9.3. Problem

A pastry shop sells cakes. On Sunday, when it sold 25 small, 15 medium and 10 large cakes, revenue was $1035. Corresponding data on Monday was 16 small; 10 medium; 4 large; $600. On Tuesday: 19 small; 11 medium; 7 large; $759. Determine the price of each kind of cake.

Solution

Let x be a price of the small cake, y is the medium cake, z is the big cake. First, we calculate the determinant of the coefficient matrix of the system

$$\begin{cases} 25x + 15y + 10z = 1035, \\ 16x + 10y + 4z = 600, \\ 19x + 11y + 7z = 759, \end{cases} \qquad (9.3)$$

using a decomposition by the first column:

$$\Delta = \begin{vmatrix} 25 & 15 & 10 \\ 16 & 10 & 4 \\ 19 & 11 & 7 \end{vmatrix} = 25\begin{vmatrix} 10 & 4 \\ 11 & 7 \end{vmatrix} - 16\begin{vmatrix} 15 & 10 \\ 11 & 7 \end{vmatrix} +$$

$$+19\begin{vmatrix} 15 & 10 \\ 10 & 4 \end{vmatrix} = 25(10\cdot 7 - 11\cdot 4) - 16(15\cdot 7 - 11\cdot 10) +$$

$$+19(15\cdot 4 - 10\cdot 10) = 25\cdot 26 - 16\cdot(-5) + 19\cdot(-40) = -30.$$

Replacing the first column of the coefficient matrix by the right-hand side, we obtain

$$\Delta_x = \begin{vmatrix} 1035 & 15 & 10 \\ 600 & 10 & 4 \\ 759 & 11 & 7 \end{vmatrix} = 1035\begin{vmatrix} 10 & 4 \\ 11 & 7 \end{vmatrix} - 600\begin{vmatrix} 15 & 10 \\ 11 & 7 \end{vmatrix} +$$

$$+759\begin{vmatrix} 15 & 10 \\ 10 & 4 \end{vmatrix}.$$

Noticing that all the determinants of the second order were calculated early, we find that

$\Delta_x = 1035\cdot 26 - 600\cdot(-5) + 759\cdot(-40) = = -450.$

Therefore, by Cramer's rule $x = \Delta_x/\Delta = -450/(-30) = 15$.

Now, substituting the found value $x = 15$, from the first and second equation of (9.3) we obtain

$$\begin{cases} 15y + 10z = 1035 - 25\cdot 15, \\ 10y + 4z = 600 - 16\cdot 15, \end{cases} \Leftrightarrow \begin{cases} 15y + 10z = 660, \\ 10y + 4z = 360. \end{cases}$$

The determinant of the coefficient matrix for this system was calculated previously: $\Delta_{(1)} = 15\cdot 4 - 10\cdot 10 = -40$.

Accordingly, $\Delta_{(1)y} = 660\cdot 4 - 360\cdot 10 = -960$, and therefore, $y = -960/(-40) = 24$. Next, using the fact that $y = 24$, the equation $15y + 10z = 660$ defines $10z = 660 - 15\cdot 24 = 300$.

So, we found that a small cake cost \$15, medium cake cost \$24, and large cake cost \$30.

Note, that the solution process will be easier if you bring in another element of Gauss elimination method into Cramer’s rule: first simplify the system — get zero in the second or third column of the systems’ coefficient matrix.

For this, in system (9.3) we divide the first equation by 5, second equation by 2, subtract the results and obtain the system:

$$\begin{cases} 5x + 3y + 2z = 207, \\ 3x + 2y + 0z = 93, \\ 19x + 11y + 7z = 759, \end{cases} \qquad (9.4)$$

We compute the determinant of the system (9.4):

$$\Delta = \begin{vmatrix} 5 & 3 & 2 \\ 3 & 2 & 0 \\ 19 & 11 & 7 \end{vmatrix} = 5\begin{vmatrix} 2 & 0 \\ 11 & 7 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 11 & 7 \end{vmatrix} + 19\begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} =$$

$= 5(2\cdot 7 - 11\cdot 0) - 3(3\cdot 7 - 11\cdot 2) + 19(3\cdot 0 - 2\cdot 2) =$

$= 5 \cdot 14 - 3 \cdot (-1) + 19 \cdot (-4) = -3.$

Replacing the first column of the coefficient matrix by the right–hand side, and noticing that all the determinants of the second order were calculated early, we find that

$\Delta_x = 207 \cdot 14 - 93 \cdot (-1) + 759 \cdot (-4) = -45.$ Then,

$x = \Delta_x / \Delta = -45/(-3) = 15.$

Now, substitute the found value $x = 15$ in the second equation of the system (9.4): $3 \cdot 15 + 2y = 93.$

Hence, $2y = 48 \Rightarrow y = 24.$

Then, using the found values from the equation

$5x + 3y + 2z = 207,$ we determine

$z = (207 - 5 \cdot 15 - 3 \cdot 24)/2 = 30.$

Exercises 9.3.

C. The workshop makes 3 kinds of lusters. One luster of the 1st kind requires 2 hours on assembly, 0.1 hours on check, 0.5 hours on packing. The appropriate data on the second kind of luster 2.5 hours, 0.2 hours and 1 hour; till the third: 3 hours, 0.2 hours and 0.75 hours. How much lusters are made, if on assembly 570 hours, on check 40 hours, on packing 180 hours are used?

H. Three farmers have decided to spend general price politics. What prices should be established on meat, carrots and potato to compensate all costs, if it is known: the 1st farmer has made 2 tons of meats, 30 tons of carrots and 20 tons of potato, having spent $15 000; for the 2nd farmer — 1; 10; 50; 12500 and for the 3rd farmer— 2; 25; 30; 15500 accordingly?

9.4. Problem

Famous glutton Robin Bobbing Bareback overfed cakes. Then it was counted up that he has used 1296 units of carbohydrates, 1235 units of fibers and 1086 units of fats. How many cakes of each kind he ate, if the cake A contains 37 units of carbohydrates, 25 units of fibers and 32 units of *fats, cake B contains 40 units of carbohydrates, 40 units of fibers and 32 units of fats, cake C contains 17 units of carbohydrates, 20 units of fibers and 13 units of fats?*

Solution

Enter the appropriate notations, we obtain the system

$$\begin{cases} 37a + 40b + 17c = 1296, \\ 25a + 40b + 20c = 1235, \\ 32a + 32b + 13c = 1086. \end{cases} \qquad (9.5)$$

The solving of the system (9.5) we begin from an intermediate step — comfortably make zero one of coefficients at *b* or *c*. For the given system the simplest variant is to subtract the 2^{nd} equation from the 1^{st}.

After that, it is useful to divide the coefficients of the second equation by 5:

$$\begin{cases} 12a - 3c = 61, \\ 25a + 40b + 20c = 1235, \\ 32a + 32b + 13c = 1086, \end{cases} \Rightarrow \begin{cases} 12a - 3c = 61, \\ 5a + 8b + 4c = 247, \\ 32a + 32b + 13c = 1086. \end{cases}$$

The determinant Δ of coefficient matrix:

$$\begin{vmatrix} 12 & 0 & -3 \\ 5 & 8 & 4 \\ 32 & 32 & 13 \end{vmatrix} = 12\begin{vmatrix} 8 & 4 \\ 32 & 13 \end{vmatrix} - 5\begin{vmatrix} 0 & -3 \\ 32 & 13 \end{vmatrix} + 32\begin{vmatrix} 0 & -3 \\ 8 & 4 \end{vmatrix} =$$

$$=12\cdot(-24) - 5\cdot 96 + 32\cdot 24 = 0.$$

Replacing the first column of coefficient matrix with the right–hand side, we calculate the determinant Δ_a. Notice, that all determinants of the 2^{nd} order necessary for calculation Δ_a, are already known:

$\Delta_a = 61\cdot(-24) - 247\cdot 96 + 1086\cdot 24 = 888.$ As Δ_a is nonzero and $\Delta = 0$, then system has no solution. It means there is some mistake in the problems data.

Exercises 9.4.

C. According to Islom's data, on the first day, he sold 8 kilograms of sweets, 5 kg of biscuits, 7 kg of flour and earned 3101 soms. On the second day, the corresponding data was obtained: 2 kg; 7 kg: 9 kg; 2187 soms. On the third day: 14 kg; 3 kg; 5 kg; 4017 soms. Determine the prices of these products.

H. Solve the system: $\begin{cases} 17x - 11y + 32z = 10, \\ 21x + 5y + 51z = 14, \\ 13x - 27y + 13z = 5. \end{cases}$

9.5. Problem

It is said that progenies of Gabriel Cramer (July 31, 1704 – January 4, 1752), upon learning about a new method for solving systems of linear algebraic equations, wrote a letter to Carl Friedrich Gauss (April 30, 1777 – February 23, 1855). In this letter they invited him to come for vacation to Switzerland together with colleagues, in order to discuss the merits of his method in a pleasant atmosphere. In response, Gauss wrote that he would be pleased to come. In their small delegation there would be x women, y children, z men and

$$\begin{cases} 15x - 18y - 14z = -63, \\ 25x + 27y - 17z = 104, \\ 175x + 18y - 138z = 101. \end{cases} \qquad (9.6)$$

Arriving at the appointed time, Gauss and his colleagues were pleasantly surprised that the desired number of places to stay was prepared.

Later, it turned out that, it was first suggested to professor Orthodoxkramer to solve the system (9.6).

He calculated the determinant of the matrix of coefficients:

$$\Delta = \begin{vmatrix} 15 & -18 & -14 \\ 25 & 27 & -17 \\ 175 & 18 & -138 \end{vmatrix} = 15\begin{vmatrix} 27 & -17 \\ 18 & -138 \end{vmatrix} -$$

$$-25\begin{vmatrix} -18 & -14 \\ 18 & -138 \end{vmatrix} + 175\begin{vmatrix} -18 & -14 \\ 27 & -17 \end{vmatrix} =$$

$$= 15\cdot(-3420) - 25\cdot 2736 + 175\cdot 684 = 0,$$

and said it was impossible to determine the number of guests because determinant is zero.

After that a scientific seminar was arranged, where the system was examined by joint efforts.

Since the first and the third equations of the system (9.6) had term $18y$, to simplify the system, to the third equation was added the first, and the result was divided by 38. They obtained the system

$$\begin{cases} 15x - 18y - 14z = -63, \\ 25x + 27y - 17z = 104 \\ 5x - 4z = 1. \end{cases} \qquad (9.7)$$

The determinant of the coefficient matrix of system (9.7) was of course also equal to zero:

$$\Delta = \begin{vmatrix} 15 & -18 & -14 \\ 25 & 27 & -17 \\ 5 & 0 & -4 \end{vmatrix} = 15 \begin{vmatrix} 27 & -17 \\ 0 & -4 \end{vmatrix} -$$

$$-25 \begin{vmatrix} -18 & -14 \\ 0 & -4 \end{vmatrix} + 5 \begin{vmatrix} -18 & -14 \\ 27 & -17 \end{vmatrix} =$$

$$= 15\cdot(-108) - 25\cdot 72 + 5\cdot 684 = 0,$$

After long deliberation and debate, someone remembered that Cramer himself wrote formula $x = \Delta_x/\Delta$ as $x\cdot\Delta = \Delta_x$. This was a decisive moment.

Calculating the determinant Δ_x:

$$\Delta_x = \begin{vmatrix} -63 & -18 & -14 \\ 104 & 27 & -17 \\ 1 & 0 & -4 \end{vmatrix} = -63 \begin{vmatrix} 27 & -17 \\ 0 & -4 \end{vmatrix} -$$

$$-104 \begin{vmatrix} -18 & -14 \\ 0 & -4 \end{vmatrix} + 1 \begin{vmatrix} -18 & -14 \\ 27 & -17 \end{vmatrix} =$$

$$= -63\cdot(-108) - 104\cdot 72 + 1\cdot 684 = 0,$$

they saw that unknown x may be any number.

Next, setting $x = p$, from the 3rd equation of system (9.7), they got $z = (5p - 1)/4$. Then, from the first equation they obtained $y = (-5p + 133)/36$.

So it turned out that system (9.7) has infinitely many solutions $x = p$; $y = (-5p + 133)/36$; $z = (5p - 1)/4)$, where p is any number.

Now they had to specify the values of unknowns. From non-negativity and integer condition of unknowns of the system: $(p \geq 0; \quad (-5p + 133)/36 \geq 0; \quad (5p - 1)/4 \geq 0) \Rightarrow$

$(p \geq 0;\ 26.6 \geq p;\ p \geq 0.25) \Rightarrow\ p = 1,\ 2,\ \ldots 26.$
Further, the condition on the variable z: $z = (5p - 1)/4$, left the values $1;\ 5;\ 9;\ 13;\ 17;\ 21;\ 25$.
Finally, the condition on y: $y = (-5p + 133)/36$ showed that the value p may be only 5.
Therefore, $x = 5;\ \ y = (-5\cdot 5 + 133)/36 = 3;$
$z = (5\cdot 5 - 1)/4 = 6.$
"That's how we found out that there are going to be the five women, three children and six men", — finished the hosts.

Exercises 9.5.

Solve the systems:

C. $\begin{cases} 7x - 3y + 3z = 10, \\ 11x + 5y + 21z = 14, \\ 38x - 26y = 56. \end{cases}$ **H.** $\begin{cases} 17x - 11y + 2z = 10, \\ 23x + 11y + 51z = 14, \\ 40x + 53z = 24. \end{cases}$

9.6. Problem

Investigate for solvability $\begin{cases} 2x + 3y + (1 + a)z = 2, \\ ax + 5y + 4z = 3, \\ 7x + 2y - 2z = 2. \end{cases}$

Solution

We can use different methods. Let use the Cramer-Gauss method. In order to solve the problem, we calculate the determinants Δ and Δ_x:

$$\Delta = \begin{vmatrix} 2 & 3 & 1 + a \\ a & 5 & 4 \\ 7 & 2 & -2 \end{vmatrix} = 2(-18) - a(-8 - 2a) + 7(7 - 5a) =$$

$= 2a^2 - 27a + 13;$

$\Delta_x = 2(-18) - 3(-8 - 2a) + 2(7 - 5a) = -4a + 2.$

Let solve the corresponding equations:

$\Delta = 0 \Leftrightarrow 2a^2 - 27a + 13 = 0 \Rightarrow a = 0.5$ or $a = 13;$

$\Delta_x = 0 \Leftrightarrow -4a + 2 = 0 \Rightarrow a = 0.5.$

(Also check that $\Delta_y = \Delta_z = 0$ for $a = 0.5$).

– for $a \neq 0.5$ and $a \neq 13$, $\Delta \neq 0$, then the system has a unique solution;

– for $a = 13$, $\Delta = 0$ and $\Delta_x = -50 \neq 0$, then the system has no solution.
– for $a = 0.5$, $\Delta = 0$, $\Delta_x = \Delta_y = \Delta_z = 0$, then the system has infinitely many solutions.

Exercises 9.6.

Explore for solvability

C. $\begin{cases} x - ay + z = 1, \\ ax - y + z = a, \\ x + y - z = 0. \end{cases}$ **H.** $\begin{cases} x + y - z = 2, \\ x + 2y + z = 3, \\ x + y + (a^2 - 5) z = a. \end{cases}$

9.7. Problem

The wolf met Red Riding Hood at the point with coordinates (1, –1), and then they reached the grandmother's house in different ways, which is located at the point with coordinates (4, 14). The routes along which the Wolf and Little Red Riding Hood moved were parts of the corresponding parabolas.

Determine the second coordinate of the house in which Little Red Riding Hood lives, knowing that the first coordinate is 2 and that she passed through a point with coordinates (3, 5).

Determine the coordinates of the point where the Wolf was when Little Red Riding Hood passed through the point with coordinates (3, 5), knowing that he began to move from his home, located at the point with coordinates (– 3, 21).

Solution

In order to write the equation of motion of Little Red Riding Hood it's necessary to know that a parabola is given by a quadratic function of the form $y = ax^2 + bx + c$. The coefficients of this function can be determined, since there are coordinates of three points that Little Red Riding Hood visited. So, substituting the coordinates of the points *(1, –1)*, *(4, 14)* and *(3, 5)* into the function $y = ax^2 + bx + c$, we obtain the system:

$$\begin{cases} a\cdot 1^2+b\cdot 1+c=-1,\\ a\cdot 3^2+b\cdot 3+c=5,\\ a\cdot 4^2+b\cdot 4+c=14, \end{cases} \Leftrightarrow \begin{cases} a\cdot 1+b\cdot 1+c=-1,\\ a\cdot 9+b\cdot 3+c=5,\\ a\cdot 16+b\cdot 4+c=14. \end{cases}$$

In order to solve the resulting system, we calculate the determinants Δ and Δ_a:

$$\Delta=\begin{vmatrix} 1 & 1 & 1\\ 9 & 3 & 1\\ 16 & 4 & 1 \end{vmatrix} = 1(-1)-9\cdot(-3)+16\cdot(-2)=-6;$$

$\Delta_a = -1(-1)-5\cdot(-3)+14\cdot(-2)=-12.$

Hence, $a=(-12)/(-6)=2$.

Substituting $a = 2$ into the first and second equations of the system, we obtain the system of second order $\begin{cases} b\cdot 1+c=-3,\\ b\cdot 3+c=-13. \end{cases}$ Solution of this system $b=-5;\ \ c=2$.

Thus, it turned out that the movement of the Little Red Riding Hood is described by the function $y = 2x^2 - 5x + 2$. From this we get that the second coordinate of the Little Red Riding Hood's house:

$y = 2(-2)^2 - 5(-2) + 2 = 20.$

Substituting the coordinates of the points $(1, -1)$, $(4, 14)$ and $(-3, -21)$, we obtain a system that allows us to determine the corresponding function for the Wolf:

$$\begin{cases} a\cdot 1^2+b\cdot 1+c=-1,\\ a\cdot(-3)^2+b\cdot(-3)+c=-21,\\ a\cdot 4^2+b\cdot 4+c=14, \end{cases} \Leftrightarrow \begin{cases} a\cdot 1+b\cdot 1+c=-1,\\ a\cdot 9-b\cdot 3+c=-21,\\ a\cdot 16+b\cdot 4+c=14. \end{cases}$$

We solve this system by the Cramer-Gauss method also:

$$\Delta=\begin{vmatrix} 1 & 1 & 1\\ 9 & -3 & 1\\ 16 & 4 & 1 \end{vmatrix} = 1(-7)-9\cdot(-3)+16\cdot 4=84;$$

$\Delta_a = -1(-7)-(-21)\cdot(-3)+14\cdot 4=0$. Hence, $a = 0/84 = 0$.

Substituting $a = 0$ into the first and second equations of the system, we obtain the system of second order

$$\begin{cases} b \cdot 1 + c = -1, \\ -b \cdot 3 + c = -21. \end{cases}$$ Solution of this system $b = 5; \quad c = -6.$

It turned out that the Wolf was moving in a straight line determined by a linear function $y = 5x - 6$. Since the distance in a straight line is the shortest, it becomes clear why the Wolf appeared at grandmother faster than Little Red Riding Hood. Because the Wolf moved in a straight line $y = 5x - 6$, from the equality $5 \cdot 3 - 6 = 11$ we get that when Little Red Riding Hood passed through a point with coordinates $(3, 5)$, the Wolf was at a point with coordinates $(3, 11)$.

Exercises 9.7.

C. Write the equation of a parabola that goes through points $(-3, 0), (-1, -4)$ and $(2, 25)$.

H. Write the equation of a parabola that goes through points $(-1, 6), (2, 9)$ and $(3, 26)$.

Appendix

Let's describe algorithm of the solving of system n of the linear algebraic equations with n unknowns by the Cramer-Gauss method.

To solve the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n . \end{cases} \qquad (9.8)$$

It is necessary:

A. To calculate the determinant Δ of a matrix of coefficients of system (9.8) using a method of decomposition on the first column: $\Delta = a_{11}\Delta_1 - a_{21}\Delta_2 + \ldots + (-1)^{n-1}a_{n1}\Delta_n$. Here Δ_m is determinant (minor), received from Δ by deletion of the first column and row with number m, where $m = 1, 2, \ldots, n$.

To calculate Δ (1), using minors Δ_k calculated at a presence Δ: $\Delta(1) = b_1\Delta_1 - b_2\Delta_2 + \ldots + (-1)^{n-1}b_n\Delta_n$.

B. If $\Delta = 0$, $\Delta(1) \neq 0$, then the system (9.8) has no the solutions. Otherwise, we shall continue the solving, excluding the equation with number k, where Δ_k is the minor with the largest absolute value, from system (9.8) and solve the system of the order $(n - 1)$ by substituting $\Delta(1)/\Delta$ instead of x_1 if $\Delta \neq 0$ and substituting instead of x_1 the parameter p (where p can be any number) if $\Delta = 0$ and $\Delta(1) = 0$.

Note

If at some stage of the solution the determinant of the matrix of coefficients of the system is non-zero, then at subsequent stages it will be non-zero.

It is easy understand how to solve system by Cramer-Gauss method, in which the number of unknowns does not coincide with a number of the equations. If number of equations m greater, then number of unknowns n, it is enough to find a solution of the system of first n equations and then check it on other $m - n$ equations of the system. If number of equations m less, then number of unknowns n, it is enough to choose $(n - m)$ unknowns as parameters and express other unknown variables through them in the parametric form.

Summary

1. Mechanized bakery *A* sells 50 % of it's production in Bishkek, 20 % in Kant, 30 % in Tokmak. Corresponding data for mechanized bakery *B*: 40 %, 40 %, 20 %; and for mechanized bakery *C*: 30 %, 40 %, 30 %. How much is made at each factory, if in Bishkek have received 219 tons, in Kant 192 tons, in Tokmak 144 tons?

2. The shop sells only three types of confection. First contains 0.3 parts of candy, 0.3 parts of butterscotch and 0.4 parts of chocolate sweets. The second type contains 0.2; 0.3; 0.5 parts, third 0.5; 0.4; 0.1 parts accordingly. How much 1kg packages of each mix are in shop, if it is known, that there are 28 kg of candy, 28.2 kg of butterscotch, 31.8 kg of chocolate sweets?

3. Three trucks have brought flour in a warehouse in bags of three different sizes. In the first truck there were 10 bags of the 1st type, 7 bags of the 2nd and 11 of the third, a total of 1 570 kg of flour, in second – 12, 8, 5 bags and 1 290 kg, in third – 10, 5, 10 bags and 1 400 kg. How much did each bag weigh?

4. Handing over warehouses, storekeeper Dirtythief specified, that in the first warehouse there are 30 small, 15 medium, and 20 large bags of sugar. Total 2850 kg. Corresponding data for the second warehouse: 18; 22; 11; 2300 kg; and for the third warehouse: 42; 8; 29; 3100 kg.

as well as paying attention to the surname storekeeper, Sherlock Holmes, who is familiar with the theory of linear equations, found that there is a mismatch of data. How did he know that?

As a result of an additional investigation, it was found that in the third warehouse sugar was stolen from any bags, and the weight of a small bag should be 30 kg. Find how much sugar has been stolen.

5. The Gamma Company manufactures three types of lamps, labeled P, R, and Q. Each lamp processed in two departments, I and II. Time requirement and profit per unit for each lamp type is as follow:

	P	**R**	**Q**
Man-hours in I	2.1	2.9	1.2
Man-hours in II	3.9	2.2	3
Profit	$5	$4.1	$3.4

How many lamps of each type were produced, if company spent 441 man-hours in I, 567 man-hours in II and received $822 profit?

6. The Teta Company manufactures products E, F, and G. Each product processed in three departments with time requirements per unit as follow:

	E	F	G
Assembling	19	31	20
Painting	11	18	23
Finishing	21	33	9

How many product of each type were manufactured, if company spent 7570 man-hours for assembling, 5190 man-hours for painting and 7260 man-hours for finishing?

7–11. Solve the systems:

7. $\begin{cases} x + y + 3z = 10, \\ 2x + y + z = 14, \\ 3x + y - z = 19. \end{cases}$ **8**. $\begin{cases} 2x - 3y + 7z = 28, \\ 3x + 5y - 10z = -29, \\ 7x - 3z = 5. \end{cases}$

9. $\begin{cases} 3x + 2y + 4z = 23, \\ 2x + y + z = 14, \\ 5x + 2y = 33. \end{cases}$ **10**. $\begin{cases} 317x + 4y - 7z = -21, \\ 511x - 3y + 11z = 1542, \\ -263x - 7y + 5z = 584. \end{cases}$

11. $\begin{cases} 86x - 3y + 11z = 20, \\ 149x - 17z = 583, \\ -371x + 7y + 6z = -811 \end{cases}$

12. Explore for solvability $\begin{cases} ax - z = a + 1, \\ ay - 3z = a + 3, \\ 4x - az = 4 + a. \end{cases}$

13. Write the equation of a parabola that goes through points *(–2, –15)*, *(1, 0)* and *(4, 39)*.

§ 10. Matrix. Operations on matrices.

The great Central Asian poet Saadi said: "Whoever has studied the sciences and does not apply them, is like the one who plowed but does not sow." Most likely, in such situations, the fact is that the corresponding person was taught only to plow. And at the same time, he was not shown how to sow. Unfortunately, this situation often takes place in the process of teaching mathematics in general, and in training future economists in particular. In this section we try to show that the concept of a matrix arises naturally and the use of matrices significantly helps in the analysis of economic situations.

10.1. Problem

A company has two enterprises, which produce three kinds of goods. In the first half of the year, the first enterprise produced seventeen thousand units of good I, three thousand units of good II, and eleven thousand units of good III. In the second half of the year, the same enterprise produced nineteen thousand units of good I, five thousand units of good II, and eight thousand units of good III.
The second enterprise in the first half of the year produced fifteen thousand units of good I, eighteen thousand units of good II, and two thousand units of good III. Corresponding data for the second half of the year are 15000; 15000; 6000.
If the information is presented as before, it would take a lot of time and space, and it would be difficult to work with it. It is much more convenient to present the data in the form of a rectangular table. For example, activities of enterprises in the first half of the year, in thousands of units:

	Good I	**Good II**	**Good III**
Enterprise1	17	3	11
Enterprise 2	15	18	2

The Corresponding data for the second half of the year:

	Good I	Good II	Good III
Enterprise 1	19	5	8
Enterprise 2	15	15	6

In this form it will also be convenient to introduce market prices ($/unit):

	1st half	2nd half
Good I	8	7.5
Good II	17	18
Good III	23	26

and the production costs of goods (in thousands of $):

	1st half	2nd half
Enterprise 1	350	400
Enterprise 2	380	440

Adding 17 from the upper left corner of the first table to 19 from the same place of the second table we find that the first enterprise produced 36 thousand units of good I during the year. In order to get all the data for the year, we must add up remaining numbers. Therefore, it will be much more convenient if the tables are placed nearby.

These considerations lead us to the necessity of introducing the concept of a matrix and operations on matrices.

Definition

Numerical matrix is a rectangular table made up of numbers. It has dimensions – the number of rows and the number of columns, and elements are the numbers in the table.

For example, a report on the production of the first enterprise in the first half of the year may be given by (2x3) matrix – $\begin{pmatrix} 17 & 3 & 11 \\ 15 & 18 & 2 \end{pmatrix}$, i.e. the matrix with two rows and three columns. Typically, matrices are denoted by capital letters. Elements of the matrix are denoted by small letters with two indices, where the first index refers to the row number and the second index refers to the column number.

Denoting the matrix above by $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ or shorter $A = (a_{ij})$, we can say that the i-th enterprise produced a_{ij} thousand units of good number j.

Prices can be written in the form of matrix P of the order (3×2): $P = \begin{pmatrix} 8 & 7.5 \\ 17 & 18 \\ 23 & 26 \end{pmatrix}$.

If the number of rows of the matrix equals the number of columns, then we are talking about a square matrix of order n (n – the number of rows and number of columns). Thus, the costs can be expressed in a matrix of the second order:

$C = \begin{pmatrix} 350 & 400 \\ 380 & 440 \end{pmatrix}$.

Very often a matrix consisting of only one line or only one column is called a vector.

For example, the prices of the first half of the year form a matrix $P1 = \begin{pmatrix} 8 \\ 17 \\ 23 \end{pmatrix}$ corresponding to the first column of matrix P.

Denoting a matrix, which reflects the results of the second half of the year, by B and a matrix, which reflects the results of the whole year, by $A + B$ we obtain

$A = \begin{pmatrix} 17 & 3 & 11 \\ 15 & 18 & 2 \end{pmatrix}$; $B = \begin{pmatrix} 19 & 5 & 8 \\ 15 & 15 & 6 \end{pmatrix}$;

$A + B = \begin{pmatrix} 36 & 8 & 19 \\ 30 & 33 & 8 \end{pmatrix}$.

Matrix addition:

a) addition is defined only for matrices of the same size;

b) the elements of the matrix, which is the sum of two matrices, are the sums of the corresponding elements of these matrices. In other words, if $Z = X + Y$, then $z_{ij} = x_{ij} + y_{ij}$.

Matrix subtraction is almost the same:

a) subtraction is defined for matrices of the same size;

b) the elements of the matrix, which is the difference of two matrices, are the differences between the corresponding elements of these matrices: if $Z = X - Y$, then $z_{ij} = x_{ij} - y_{ij}$.
By calculating the difference

$$B - A = \begin{pmatrix} 19 & 5 & 8 \\ 15 & 15 & 6 \end{pmatrix} - \begin{pmatrix} 17 & 3 & 11 \\ 15 & 18 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & -3 \\ 0 & -3 & 4 \end{pmatrix},$$

we see that the first enterprise in the second half of the year increased production of goods 1 and 2 and decreased production of good 3 by 3 thousand units, etc.
Compare matrix of total output N (see below about this matrix) with the matrix of production costs:

$$N - C = \begin{pmatrix} 343 & 440.5 \\ 472 & 538.5 \end{pmatrix} - \begin{pmatrix} 350 & 400 \\ 380 & 440 \end{pmatrix} = \begin{pmatrix} -7 & 40.5 \\ 92 & 98.5 \end{pmatrix},$$

we can see that the first enterprise in the first half of the year was unprofitable, in the second half of the year the situation significantly improved, and the second enterprise is consistently profitable.
Assume that the management of the company plans to increase the production of all of goods at both enterprises by 10% next year. In order to obtain a matrix of the expected results, we must multiply all elements of matrix $A + B$ is by 1.1. Obviously, we denote the resulting matrix by $1.1(A + B)$ and call it the product of the number 1.1 and matrix $A+B$: $1.1(A + B) = \begin{pmatrix} 39.6 & 8.8 & 20.9 \\ 33 & 36.3 & 8.8 \end{pmatrix}$.
Thus, ***the rule for multiplication a number and a matrix***: in order to multiply a matrix by a number we have to multiply every element of the matrix by this number.

Problem

Let

$$X = \begin{pmatrix} 12 & -2 \\ 2 & -15 \\ 17 & 21 \end{pmatrix}; \; Y = \begin{pmatrix} 7 & 12 & 17 \\ 2 & -5 & -2 \end{pmatrix}; \; Z = \begin{pmatrix} 70 & -2 & 9 \\ 23 & 51 & 4 \end{pmatrix}.$$

Find: $X + Y$; $X - 5Y$; $X + 0.5Z$; $2X - Z$; $Y + Z$; $9X$; $-0.5Y$; $Y - Z$; $3Z - 2Y$.

Solution

$X+Y$; $X-5Y$; $X+0.5Z$; $2X-Z$ a not defined, because matrices have different size;

$$Y+Z = \begin{pmatrix} 77 & 10 & 26 \\ 25 & 46 & 2 \end{pmatrix}; \quad 9X = \begin{pmatrix} 108 & -18 \\ 18 & -135 \\ 153 & 189 \end{pmatrix};$$

$$-0.5Y = \begin{pmatrix} -3.5 & -6 & -8.5 \\ -1 & 2.5 & 1 \end{pmatrix}; \quad Y-Z = \begin{pmatrix} -63 & 14 & 8 \\ -21 & -56 & -6 \end{pmatrix};$$

$$3Z-2Y = \begin{pmatrix} 210 & -6 & 27 \\ 69 & 153 & 12 \end{pmatrix} - \begin{pmatrix} 14 & 24 & 34 \\ 4 & -10 & -4 \end{pmatrix} =$$

$$= \begin{pmatrix} 196 & -30 & -7 \\ 65 & 163 & 16 \end{pmatrix}.$$

Exercises 10.1.

C. Let

$$S = \begin{pmatrix} 9 & 21 & 15 \\ 22 & -7 & 11 \\ 7 & 2.3 & -4 \end{pmatrix}; T = \begin{pmatrix} 8 & 2.1 & -8 \\ -6 & 1 & 14 \\ 6 & 18 & -2 \end{pmatrix}; R = \begin{pmatrix} 12 & -2 \\ 2 & 5 \\ -17 & 1 \end{pmatrix}.$$

Find: $S+T$; $S+R$; $T-R$; $T-S$; $0.2R$; $-7T$; $3S-5T$.

H. Let

$$K = \begin{pmatrix} 21 & -2 \\ 2 & -5 \\ -1.7 & 10 \end{pmatrix}; \quad L = \begin{pmatrix} 8 & 2.1 & -8 \\ -6 & 1 & 14 \\ 6 & 18 & -2 \end{pmatrix}; \quad M = \begin{pmatrix} 2 & -2.7 \\ 32 & -5 \\ 1.7 & 1 \end{pmatrix}.$$

Find: $K+L$; $L+M$; $K+M$; $L-M$; $K-M$; $4M$; $1.1L$; $3K-2.5M$.

10.2. Matrix multiplication

The rule for multiplying a matrix by another matrix is harder. Let's start with leading considerations.

It is usually more convenient to represent the output not in units, meters or tons, but in monetary terms. Therefore, we need to know the price of each product.

To find the value of goods produced by the enterprise, we have to multiply production volumes of each good by the unit price of that good and then add these products.
Let's express the activities of the company in the first half of the year in monetary terms:

$$A \cdot P1 = \begin{pmatrix} 17 & 3 & 11 \\ 15 & 18 & 2 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 17 \\ 23 \end{pmatrix} = \begin{pmatrix} 17 \cdot 8 + 3 \cdot 17 + 11 \cdot 23 \\ 15 \cdot 8 + 18 \cdot 17 + 2 \cdot 23 \end{pmatrix} = $$

$$= \begin{pmatrix} 343 \\ 472 \end{pmatrix}$$

(recall that *P1* is the first column of matrix *P*, it represents prices of the first half of the year).
Now, expressing the activities of the company in the second half of the year the same way

$$A \cdot P2 = \begin{pmatrix} 19 & 5 & 8 \\ 15 & 15 & 6 \end{pmatrix} \cdot \begin{pmatrix} 7.5 \\ 18 \\ 26 \end{pmatrix} = \begin{pmatrix} 19 \cdot 7.5 + 5 \cdot 18 + 8 \cdot 26 \\ 15 \cdot 7.5 + 15 \cdot 18 + 6 \cdot 26 \end{pmatrix} = $$

$$= \begin{pmatrix} 440.5 \\ 538.5 \end{pmatrix},$$

we have *N* is a matrix, whose first column is the nominal value of goods produced by the enterprises in the first half of the year and the second column – the second half of the year:

$$N = \begin{pmatrix} 343 & 440.5 \\ 472 & 538.5 \end{pmatrix}.$$

As prices change over time, in order to analyze the economic situation, economists use the concepts of nominal and real value of goods.
The real value is the value of goods in prices of a certain period selected as a base period.
Nominal value is the value of goods in current prices.
For example, if prices are rising, and the volume of production does not change, the nominal value of the goods produced by an enterprise is growing, and the real value remains unchanged.

If we calculate the value of goods produced by several companies of the same type in prices of different periods of time and write the results in the form of a matrix, then this matrix is the product of matrices expressing production and prices.

For example, the product of matrix *B*, reflecting the activities of the company in the second half of the year, and matrix *P*, reflecting prices in the first and second halves of the year, is:

$$B \cdot P = \begin{pmatrix} 19 & 5 & 8 \\ 15 & 15 & 6 \end{pmatrix} \cdot \begin{pmatrix} 8 & 7.5 \\ 17 & 18 \\ 23 & 26 \end{pmatrix} =$$

$$= \begin{pmatrix} 19\cdot 8 + 5\cdot 17 + 8\cdot 23 & 19\cdot 7.5 + 5\cdot 18 + 8\cdot 26 \\ 15\cdot 8 + 15\cdot 17 + 6\cdot 23 & 15\cdot 7.5 + 15\cdot 18 + 6\cdot 26 \end{pmatrix} =$$

$$= \begin{pmatrix} 421 & 440.5 \\ 559 & 538.5 \end{pmatrix}.$$

So, if we take the prices of the first half of the year as real prices, the first enterprise in the second half produced the real value of goods equal to 421 thousand dollars, the nominal value is 440.5 thousand dollars. The second enterprise produced 559 thousand and 538.5 thousand dollars correspondingly.

Let’s formulate the rule for matrix multiplication:

There are matrices

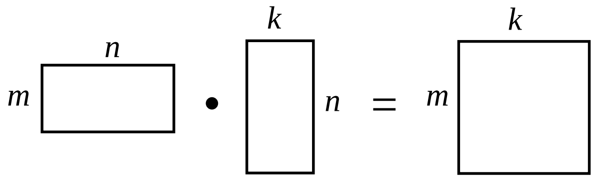
$$X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}, \qquad Y = \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1k} \\ y_{21} & y_{22} & \dots & y_{2k} \\ \dots & \dots & \dots & \dots \\ y_{n1} & y_{n2} & \dots & y_{nk} \end{pmatrix}.$$

Then their product is matrix $Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1k} \\ z_{21} & z_{22} & \dots & z_{2k} \\ \dots & \dots & \dots & \dots \\ z_{m1} & z_{m2} & \dots & z_{mk} \end{pmatrix}$,

whose elements are calculated according to the rule:

$$z_{ij} = x_{i1}y_{1j} + x_{i2}y_{2j} + \dots + x_{in}y_{nj}, \qquad i = 1, \dots, m; \; j = 1, \dots, k.$$

To memorize the rule will use the following image:



Note that the product is defined only if the number of columns of the first factor is equal to the number of rows of the second (output value can be determined only in the case where each type of product has its price).

Matrix multiplication is significantly different from the multiplication of numbers: We know that the product of numbers is subject to the commutative law of numbers — the relocation of the factors does not change the product. For matrices this law does not hold. After the relocation of the factors matrix product can be simply not defined — for example, matrix $A \cdot P1$ is calculated in the preceding paragraph, and matrix $P1 \cdot A$ is not defined as the multiplication cannot be performed. And even if *XY* and *YX* are defined, *XY* is not necessarily equal to *YX*.

Example $X \cdot Y = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 5 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 10 & -4 \\ 15 & -9 \end{pmatrix};$

$$Y \cdot X = \begin{pmatrix} 5 & -2 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 9 & -3 \end{pmatrix}.$$

Exercises 10.2.

C. Let

$$S = \begin{pmatrix} 9 & 21 & 15 \\ 22 & -7 & 11 \\ 7 & 2.3 & -4 \end{pmatrix};\ T = \begin{pmatrix} 8 & 2.1 & -8 \\ -6 & 1 & 14 \\ 6 & 18 & -2 \end{pmatrix};\ R = \begin{pmatrix} 12 & -2 \\ 2 & 5 \\ -17 & 1 \end{pmatrix}.$$

Find: $S \cdot T$; $S \cdot R$; $T \cdot S$; $R \cdot T$.

H. Let

$$K = \begin{pmatrix} 21 & -2 \\ 2 & -5 \\ -1.7 & 10 \end{pmatrix};\; L = \begin{pmatrix} 8 & 2.1 & -8 \\ -6 & 1 & 14 \\ 6 & 18 & -2 \end{pmatrix};\; M = \begin{pmatrix} 2 & -2.7 \\ 32 & -5 \\ 1.7 & 1 \end{pmatrix}.$$

Find: $K{\cdot}L$; $L{\cdot}M$; $K{\cdot}M$; $L{\cdot}K$.

10.3. Identity matrix

The main diagonal of a square matrix consists of elements, which have the same row number and column number – elements of the form a_{ii} .

A square matrix whose main diagonal consists of ones and all other elements are zeros, is called an identity matrix and is usually denoted by I. The identity matrix of different orders:

$$(1),\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \dots .$$

The validity of the name of the identity matrix is easily verified by a test: if we multiply any matrix by the identity matrix, we get the original matrix ($A{\cdot}I = A$ or $I{\cdot}A = A$):

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & 2.1 & -8 \\ -6 & 1 & 14 \\ 6 & 18 & -2 \end{pmatrix} = \begin{pmatrix} 8 & 2.1 & -8 \\ -6 & 1 & 14 \\ 6 & 18 & -2 \end{pmatrix}.$$

The fourth arithmetic operation — dividing, in the theory of matrices corresponds to a multiplication by an inverse matrix. (Rather than divide number a by number b, we can multiply a and b^{-1}, for example $7:2 = 7(2)^{-1}$).

Matrix Y is the inverse of matrix X, if $XY = I = YX$. The inverse of matrix X is usually denoted by X^{-1}.

Problem *Make sure that if* $X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *, then*

$$X^{-1} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}.$$

Solution

In order to make sure that matrix X^{-1} is indeed the inverse of matrix X, we have to multiply these matrices:

$$X X^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix} =$$

$$= \begin{pmatrix} 1\cdot(-2) + 2\cdot 1.5 & 1\cdot 1 + 2\cdot(-0.5) \\ 3\cdot(-2) + 4\cdot 1.5 & 3\cdot 1 + 4\cdot(-0.5) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows from the definition that the inverse exists only for square matrices. We will discuss the methods for finding the inverse matrix in the following sections.

Exercises 10.3. Make sure that:

C. a) if $X = \begin{pmatrix} -20 & 5 \\ -8 & 3 \end{pmatrix}$, then $X^{-1} = \begin{pmatrix} -0.15 & 0.25 \\ -0.4 & 1 \end{pmatrix}$;

b) if $T = \begin{pmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{pmatrix}$, then $T^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 3 & 2 \end{pmatrix}$.

H. a) if $X = \begin{pmatrix} 8 & -12 \\ 3 & 8 \end{pmatrix}$, then $X^{-1} = \begin{pmatrix} 0.08 & 0.12 \\ -0.03 & 0.08 \end{pmatrix}$;

b) if $T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$, then $T^{-1} = \begin{pmatrix} 1.4 & 0.2 & -0.4 \\ 1.2 & -0.4 & -0.2 \\ -1.6 & 0.2 & 0.6 \end{pmatrix}$.

10.4. Transposed matrix

To conclude this section we present the definition of the transposed matrix. If we swap rows and columns in matrix X, we get a transposed matrix denoted by X^T.

Example If $X = \begin{pmatrix} 2 & -2 \\ 2 & -5 \\ 7 & -2 \end{pmatrix}$, then $X^T = \begin{pmatrix} 2 & 2 & 7 \\ -2 & -5 & -2 \end{pmatrix}$.

Exercises 10.4.
C. Let

$$S=\begin{pmatrix} 9 & 21 & 15 \\ 22 & -7 & 11 \\ 7 & 2.3 & -4 \end{pmatrix};\ T=\begin{pmatrix} 8 & 2.1 & -8 \\ -6 & 1 & 14 \\ 6 & 18 & -2 \end{pmatrix};\ R=\begin{pmatrix} 12 & -2 \\ 2 & 5 \\ -17 & 1 \end{pmatrix}.$$

Find: S^T; R^T; $P{\cdot}R^T$; $10P + S^T$; $R^T{\cdot}P$.
H. Let

$$K=\begin{pmatrix} 21 & -2 \\ 2 & -5 \\ -1.7 & 10 \end{pmatrix};\ L=\begin{pmatrix} 8 & 2.1 & -8 \\ -6 & 1 & 14 \\ 6 & 18 & -2 \end{pmatrix};\ M\ \begin{pmatrix} 2 & -2.7 \\ 32 & -5 \\ 1.7 & 1 \end{pmatrix}.$$

Find: K^T; L^T; $L^T{\cdot}M$; $K^T - 2M$; $K^T - 2M^T$.

10.5. Problem

Ugur and Uraz trade balls. In the past period, Ugur bought 25 soccer balls, 12 basketball balls and sold 28 soccer balls, 10 basketball balls. Uraz bought 18 soccer balls, 16 basketball balls and sold 17 soccer balls, 17 basketball balls. They use a 20% mark-up — each unit of goods they sell is 20% more expensive than they buy. Determine the profit of Ugur and the profit of Uraz, knowing that they bought a soccer ball for \$8, a basketball for \$7.5. What would be their profit if a soccer ball was bought for \$7, a basketball for \$10?

Solution

Let's start by calculating revenue, costs and profit.

The Ugur's costs: $25{\cdot}8 + 12{\cdot}7.5 = 290;$
revenue: $28{\cdot}(1.2{\cdot}8) + 10{\cdot}(1.2{\cdot}7.5) = 358.8;$
profit: $358.8 - 290 = 68.8.$

The Uraz's costs: $18{\cdot}8 + 16{\cdot}7.5 = 264;$
revenue: $17{\cdot}(1.2{\cdot}8) + 17{\cdot}(1.2{\cdot}7.5) = 316.2;$
profit: $316.2 - 264 = 52.2.$

If a soccer ball was bought for \$7, a basketball ball for \$10,
the Ugur's costs: $25{\cdot}7 + 12{\cdot}10 = 295;$
revenue: $28{\cdot}(1.2{\cdot}7) + 10{\cdot}(1.2{\cdot}10) = 355.2;$

profit: *355.2 – 295 = 60.2;*
the Uraz's costs: *18·7 + 16·10 = 286;*
revenue: *17·(1.2·7) + 17·(1.2·10) = 346.8;*
profit: *316.2 – 264 = 60.8.*

The solution is complete. But, if you use matrices, the record will be noticeably more compact, and the amount of computation is less.

So, the purchase matrix and the sales matrix of Ugur and Uraz are written in the form:

$\begin{pmatrix} 25 & 12 \\ 18 & 16 \end{pmatrix}$, $\begin{pmatrix} 28 & 10 \\ 17 & 17 \end{pmatrix}$, prices matrix: $\begin{pmatrix} 8 & 7 \\ 7.5 & 10 \end{pmatrix}$.

Then, the profit of each at the corresponding prices will be expressed by the matrix expression

$$\begin{pmatrix} 28 & 10 \\ 17 & 17 \end{pmatrix} \cdot 1.2 \begin{pmatrix} 8 & 7 \\ 7.5 & 10 \end{pmatrix} - \begin{pmatrix} 25 & 12 \\ 18 & 16 \end{pmatrix} \cdot \begin{pmatrix} 8 & 7 \\ 7.5 & 10 \end{pmatrix}.$$

Note that multiplying by 1.2 is determined by using the mark–up premium of 20%. Take the price matrix out of brackets and multiply the matrix by a number:

$$\begin{pmatrix} 28 & 10 \\ 17 & 17 \end{pmatrix} \cdot 1.2 \begin{pmatrix} 8 & 7 \\ 7.5 & 10 \end{pmatrix} - \begin{pmatrix} 25 & 12 \\ 18 & 16 \end{pmatrix} \cdot \begin{pmatrix} 8 & 7 \\ 7.5 & 10 \end{pmatrix} =$$

$$= \left[\begin{pmatrix} 28 & 10 \\ 17 & 17 \end{pmatrix} \cdot 1.2 - \begin{pmatrix} 25 & 12 \\ 18 & 16 \end{pmatrix}\right] \cdot \begin{pmatrix} 8 & 7 \\ 7.5 & 10 \end{pmatrix}.$$

As a result, it turned out that in this situation, to calculate the profit, it is enough to apply the trading mark-up to the sales matrix, subtract the purchase matrix from the product and multiply the resulting difference by the price matrix.

In the case of Ugur and Uraz, we obtain

$$\left[\begin{pmatrix} 28 & 10 \\ 17 & 17 \end{pmatrix} \cdot 1.2 - \begin{pmatrix} 25 & 12 \\ 18 & 16 \end{pmatrix}\right] \cdot \begin{pmatrix} 8 & 7 \\ 7.5 & 10 \end{pmatrix} =$$

$$= \begin{pmatrix} 8.6 & 0 \\ 2.4 & 4.4 \end{pmatrix} \cdot \begin{pmatrix} 8 & 7 \\ 7.5 & 10 \end{pmatrix} = \begin{pmatrix} 68.8 & 60.2 \\ 52.2 & 60.8 \end{pmatrix}.$$

Exercises 10.5.

C. Olga, Elena and Elmira sell 2 types of caps. They use a 40% mark–up — they sell each unit of goods at 40% more expensive than they buy. Olga sold 15 caps of type A, 12 — of type B. At the same time, she bought 14 caps of type A, 16 — type B. Corresponding data from Elena: sale 14; 18, purchase 18; 17. For Elmira: sale of 25; 20, purchase 18; 28. Determine the profit of Olga, profit of Elena and profit of Elmira, knowing that they bought a cap of type A for $6, a cap of type B for $4.1. What would be their profit if a cap of type A was bought for $7, a cap of type B for $5?

H. Taalai and Bakyt sell 3 types of chess boards. They use a 25% mark-up — they sell each unit of goods at 25% more expensive than they buy. Taalai sold 11 chess boards of type A, 12 — of type B, 18 — of type C. At the same time, he bought 16 chess boards of type A, 7 — type B, 14 — type C. Corresponding data from Bakyt: sale 18; 15; 3, purchase 22; 13; 9.

Determine the profit of Taalai and profit of Bakyt, knowing that they bought a chess board of type A for $6, a chess board of type B for $4, a chess board of type C for $10. What would be their profit if a chess board of type A was bought for $10, a chess board of type B for $5, a chess board of type C for $2?

10.6. Problem

Kamilla, Nargiza and Deniza sell 3 type of calculators and they use a 20% retail margin (each unit of the goods they sell 20% more expensive than buying.) At some moment they found that Camila has sold 9 calculators of type A, 21 – of type B, 15 – of type C. At the same time, she bought 8 calculators of type A, 21 – of type B,18 – of type C. For Nargiza: sale – 22; 17; 11, purchase – 26; 11; 14 calculators correspondingly. For Denise: sale – 17; 23; 14, *purchase – 6; 18; 21 calculators correspondingly. Determine at what price are they buy calculators if it is known that Camila's profit is $ 18.9, Nargiza's profit is $20.7, Denise's profit is $ 46.2.*

Solution

We write out the sales matrix *S* and the purchase matrix *T*:

$$S = \begin{pmatrix} 9 & 21 & 15 \\ 22 & 17 & 11 \\ 17 & 23 & 14 \end{pmatrix}; \quad T = \begin{pmatrix} 8 & 21 & 18 \\ 26 & 11 & 14 \\ 6 & 18 & 21 \end{pmatrix}.$$

Considering the mark–up, we calculate the matrix (*1.2S – T*) is the matrix of the system of linear algebraic equations, which allows to determine the prices:

$$1.2 \cdot S - T = 1.2\begin{pmatrix} 9 & 21 & 15 \\ 22 & 17 & 11 \\ 17 & 23 & 14 \end{pmatrix} - \begin{pmatrix} 8 & 21 & 18 \\ 26 & 11 & 14 \\ 6 & 18 & 21 \end{pmatrix} =$$

$$= \begin{pmatrix} 2.8 & 4.2 & 0 \\ 0.4 & 9.4 & -0.8 \\ 14.4 & 9.6 & -4.2 \end{pmatrix}.$$

So, in order to determine the prices of calculators, we solve a system of linear algebraic equations:

$$\begin{cases} 2.8a + 4.2b = 18.9, \\ 0.4a + 9.4b - 0.8c = 20.7, \\ 14.4a + 9.6b - 4.2c = 46.2. \end{cases}$$

We begin the solution of the system by calculating the determinant of the matrix of coefficients by expanding along the first column:

$$\Delta = \begin{vmatrix} 2.8 & 4.2 & 0 \\ 0.4 & 9.4 & -0.8 \\ 14.4 & 9.6 & -4.2 \end{vmatrix} = 2.8[9.4\cdot(-4.2) - 9.6\cdot(-0.8)] -$$

$$-0.4\cdot[4.2\cdot(-4.2) - 9.6\cdot 0] + 14.4[4.2\cdot(-0.8) - 9.4\cdot 0] =$$

$$= 2.8\cdot(-31.8) - 0.4\cdot(-17.64) + 14.4\cdot(-3.36) = -130.368.$$

Of course, a third order determinant can be calculated in many different ways. But, the method we use – decomposition by the first column, can significantly reduce the amount of computation. So, when calculating the determinant Δ_a, we can take advantage of the fact that all the necessary minors are already calculated:

$$\Delta_a = \begin{vmatrix} 18.9 & 4.2 & 0 \\ 20.7 & 9.4 & -0.8 \\ 46.2 & 9.6 & -4.2 \end{vmatrix} =$$

$$= 18.9\cdot(-31.8) - 20.7\cdot(-17.64) + 46.2\cdot(-3.36) = -391.104.$$

Further, following the Cramer method, we must calculate the values of the determinants Δ_b and Δ_c.

But, it is much easier to move on to the exclusion of the unknown. Since $\Delta\cdot a = \Delta_a$, $a = -391.104/(-130.368) = 3$.

Further, substituting the value $a = 3$ in the first equation, we obtain: $4.2b = 18.9 - 2.8\cdot 3 = 10 \Leftrightarrow b = 2.5$.

Then, from the second equation:

$-0.8c = 20.7 - 0.4\cdot 3 - 9.4\cdot 2 \Leftrightarrow c = -4/(-0.8) = 5.$

So, we finished solving the problem, finding out that a calculator of type A costs \$3, type B costs \$2.5, type C costs \$5.

Exercises 10.6.

C. Olga, Stas and Zhibek sell 3 types of T-shirts. They use a 40% mark-up — they sell each unit of goods at 40% more expensive than they buy. Olga sold 15 T-shirts of type A, 17 — of type B, 13 — of type C. At the same time, she bought 14 T-shirts of type A, 21 — of type B, 16 —of type C. Corresponding data from Stas: sale 19; 14; 18, purchase 30; 18; 27. From Zhibek: sale of 25; 20; 16, purchase 31; 28; 25.

Determine at what prices they buy T-shirts, knowing that Olga's profit was \$72.9, Stas's (– \$23.9), Zhibek's \$16.3.

H. Tahir, Argen and Eleonora sell 3 types of pants. They use a 30% mark-up — they sell each unit of goods at 30% more expensive than they buy. Tahir sold 11 pants of type A, 23 — of type B, 28 — of type C. At the same time, he bought 16 pants of type A, 27 — of type B, 40 —of type C. Corresponding data from Argen: sale 18; 25; 30, purchase 22; 35; 39. And from Eleonora: sale of 19; 16; 26, purchase 27; 23; 31. Determine at what prices they buy pants, knowing that Tahir's profit was (–\$35.8), Argen's (–\$17), Eleonora's (–\$43.6).

Summary

1. There are matrices:

$$A = \begin{pmatrix} -3 & 5 \\ -0.2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2.1 & 5 \\ 0 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1/3 \\ 2 \end{pmatrix}.$$

Find: *a) A+B; b) 2B – 3A; c) 2B – C; d) AB; e) BA; f) A(2C); g) CB; h) BC – 3C; i) (B–3I)C*, where *I* is an identity matrix, and compare resuts with h); *j)* A^T; *k)* C^T.

2. Let $A = (3 \quad 0.4 \quad 2)$; $B = \begin{pmatrix} -1 & 2 & 7 \\ 11 & 9 & 5 \\ 10 & 0 & -4 \end{pmatrix}$;

$$C = \begin{pmatrix} -\frac{2}{3} & 4 & 9 \\ -16 & 2 & 3 \\ 8.12 & 5 & 8 \end{pmatrix}; \quad D = \begin{pmatrix} 16 \\ -4 \\ 21 \end{pmatrix}.$$

Find: *a)B + 2C; b)C – 2D; c)3A+5D; d)A – B; e)B – A, f)C – D; g) AB – CD;* h) A^T; i) C^T.

3. Let $A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $B = (0 \quad 3)$, $C = \begin{pmatrix} 4 & 10 \\ 1 & -2 \end{pmatrix}$.

Calculate: *a) AB; b) BC; c) CB; d) BC–2B.*

Check the equality $BC - 2B = B(C - 2I)$, where I – identity matrix.

4. Let $A=(5 \quad 7)$; $B = \begin{pmatrix} -2 & 8 \\ 4 & -1 \end{pmatrix}$; $C = \begin{pmatrix} 8 & -9 & 4 \\ 11 & 3 & -5 \end{pmatrix}$.

Calculate: AB; AC; BI; BA; CA; IB; $BC - 2C$.

5. The first truck transported 23 small, 19 medium and 10 large bags of flour. The second truck transported 12 small, 25 medium and 16 large bags of flour. How many kilograms of flour each truck transported , if the weight of a small bag is 24 kg, of a medium bag is 48 kg, of a large bag is 60 kg? Solve the problem after writing the information about bags and weights of bags in the form of matrices.

6. The firm ASK has three one-profiled companies, on which two types of goods are produced. The first company in the first quarter produced 12 units of the type A of product and 33 of type B. In the second quarter on the same company 16.4 units of the type A of product and 35 units of the type B of product were produced. The corresponding information for the 4th quarter represented by numbers 17 and 31. On the 2nd company in the first quarter 15.2 units of type A of product and 18 units of the type B were made. The corresponding information for the second and fourth quarters is as follows: 15, 15 and 12, 18.

The results of the 3rd company in the first, second and fourth quarters are as follows: 20; 7 and 20; 8 and 22; 8.

In addition, the annual results are known.

The first company produced 67.5 units of the type A of product, and 120 units of the type B of product. The second: 59 and 70, third: 85 and 31.

Write out the given information in the form of matrices and find the quantity of production of each type of product in the first half-year and in the third quarter.

The firm's leaders plan to increase output at each company for each type of product by 20% for the next year, clarify plans by writing them in terms of units of the goods. If the unit price of the type A of product is 120 soms, and the unit price of the type B of product is 200 soms, then for how many soms were produced goods by each company

a) in the first quarter, b) for the second half-year?

7. Make sure that:

a) if $X = \begin{pmatrix} 2 & 0.5 \\ -8 & 3 \end{pmatrix}$, then $X^{-1} = \begin{pmatrix} 0.3 & -0.05 \\ 0.8 & 0.2 \end{pmatrix}$;

b) if $T = \begin{pmatrix} -1.6 & 0.6 & 0.8 \\ 3.8 & -0.8 & -1.4 \\ -1.2 & 0.2 & 0.6 \end{pmatrix}$, then $T^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & -4 \\ 1 & 2 & 5 \end{pmatrix}$.

9. Ruslan and Mira sell 2 types of calculators. They use a 30% mark–up — they sell each unit of goods at 30% more expensive than they buy. Ruslan sold 20 calculators of type A, 35 — of type B. At the same time, he bought 23 calculators of type A, 32 — of type B. Corresponding data from Mira: sale 33; 40, purchase 34; 40. Determine the profit of Ruslan and profit of Mira, knowing that they bought a calculator of type A for $2, a calculator of type B for $4. What would be their profit if a calculator of type A was bought for $3, a calculator of type B for $2.5?

10. Altynbek and Ann sell 3 types of neckties. They use a 10% mark–up — they sell each unit of goods at 10% more expensive than they buy. Altynbek sold 20 neckties of type A, 29 — of type B, 30 — of type C. At the same time, he bought 22 neckties of type A, 26 — of type B, 30 —of type C. At Ann: sale of 18; 22; 30, purchase 15; 24; 28. Determine the profit of Altynbek and profit of Ann, knowing that they bought a necktie of type A for $5, a necktie of type B for $3, a necktie of type C for $4. What would be their profit if a necktie of type A was bought for $3, a necktie of type B for $2.5, a necktie of type C for $4?

§11. Inverse matrix

In his famous novel Les Miserables, Victor Hugo writes: “Only with the help of science can one realize the sublime dream of poets: the beauty of the social system. At the level that civilization has reached, accuracy is a necessary element of the beautiful; scientific thought not only helps artistic flair, but also complements it; a dream must be able to calculate. Conquering art should be supported by science, like a war horse. It is very important that this support is reliable.”

This quote is another confirmation of the importance of mathematics, the ability to calculate. But, despite this, perhaps, obvious truth, one can meet a significant number of people, even with higher education, who not only do not know how to count — they even deny this necessity. In our opinion, the origins of this amazing phenomenon are in school and university courses in mathematics, which are very often boring, difficult to digest, and unattractive.

Therefore, the revision of the content of mathematical courses, methods of their presentation is an urgent task. A significant place in the university course of mathematics is occupied by systems of linear algebraic equations. This is an interesting and important topic for math applications. But, unfortunately, in many textbooks there are no text problems illustrating their application. And as David Gilbert, a German universal mathematician, who made a significant contribution to the development of many areas of mathematics in the 19 – 20 century, who in the 1910 – 1920s (after the death of Henri Poincaré) was the recognized world leader of mathematicians: “In order to make the lesson is interesting, you need to look for a suitable example." This section provides examples of tasks that can make the process of learning the inverse matrix fun and useful.

11.1. Problem

Three friends Bakai, Sanjar and Samat came back for summer break from different universities. They decided to shoot balls at the basket.
Bakai has shot 13 times from under the basket, 6 times from the middle distance and 4 times from the long distance. Sanjar has shot 14 times from under the basket, 8 times from the middle distance and 2 times from the long distance. Samat has shot 16 times from under the basket, 6 times from the middle distance and 3 times from the long distance.
After competition Bakai said that according to the rules of his university he won because he scored 41 points,
Sanjar — 38 points, Samat — 40 points.
Sanjar said that according to the rules of his university he won because he scored 124 points,
Bakai — 123 points, Samat — 123 points.
Samat said that according to the rules of his university he won because he scored 87 points,
Bakai — 83 points, Sanjar — 84 points.
Kyzjibek who was present at the competition said if they used her rules they would score 80 points each.
Find out each university's rules for calculating scores and Kyzjibek's rules.

Solution

In order to determine the rules by which points are counted at Bakai's University we need to solve a system

$$\begin{cases} 13x + 6y + 4z = 41, \\ 14x + 8y + 2z = 38, \\ 16x + 6y + 3z = 40. \end{cases} \qquad (11.1)$$

Accordingly, to determine the rules of scoring at Sanjar's University we need to solve a system, replacing the right-hand side of system (11.1) — column matrix $F_1 = \begin{pmatrix} 41 \\ 38 \\ 40 \end{pmatrix}$ with $F_2 = \begin{pmatrix} 123 \\ 124 \\ 123 \end{pmatrix}$; for Samat's University we replace F_1 with $F_3 = \begin{pmatrix} 83 \\ 84 \\ 87 \end{pmatrix}$; for Kyzjibek — F_1 with $F_4 = \begin{pmatrix} 80 \\ 80 \\ 80 \end{pmatrix}$.

Instead of solving four systems, we can use an inverse matrix.

Recall that a square matrix whose main diagonal consists of ones and all other elements are zeros is called an identity matrix and it is denoted by I.

It is easy to verify that in accordance with its name the product of the identity matrix and any matrix M of the corresponding order does not change the matrix M.

If equalities $BA = I = AB$ hold, matrix B is the inverse of matrix A and is denoted by A^{-1}.

We write system (11.1) in the form of a matrix equation

$$AX = F_1, \qquad (11.2)$$

where

$$A = \begin{pmatrix} 13 & 6 & 4 \\ 14 & 8 & 2 \\ 16 & 6 & 3 \end{pmatrix}; \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; \quad F_1 = \begin{pmatrix} 41 \\ 38 \\ 40 \end{pmatrix}.$$

After determining the inverse matrix, we can multiply the equation $AX = F_1$ by the inverse matrix A^{-1}: $A^{-1}AX = A^{-1}F_1$, and since $A^{-1}AX = X$, the equality $X = A^{-1}F_1$ allows us to understand the rules of scoring at Bakai's University.

Correspondingly, the rules of scoring at Sanjar's University will be obtained as the product $A^{-1}F_2$; at Samat's University — as $A^{-1}F_3$; and according to Kyzjibek — as $A^{-1}F_4$.

From the equation $A\,A^{-1} = I$ and the rules of matrix multiplication, it follows that: the 1st column of the inverse matrix is a solution of the system $AX = I_1$, where I_1 is the 1st

column of the identity matrix; the 2nd column of the inverse matrix is a solution of the system $AX = I_2$, where I_2 is the 2nd column of the identity matrix; etc.
Therefore, in order to find the 1st column of the matrix inverse to the matrix $A = \begin{pmatrix} 13 & 6 & 4 \\ 14 & 8 & 2 \\ 16 & 6 & 3 \end{pmatrix}$,
it is necessary to solve the system

$$\begin{cases} 13x + 6y + 4z = 1, \\ 14x + 8y + 2z = 0, \\ 16x + 6y + 3z = 0. \end{cases} \qquad (11.3)$$

We should say that to solve the system, we could use any method. Now we show how to find the inverse matrix using the Cramer-Gauss method.
The determinant of A is equal to:
$\Delta = 13(8\cdot3 - 6\cdot2) - 14(6\cdot3 - 6\cdot4) + 16(6\cdot2 - 8\cdot4) =$
$=13\cdot12 - 14(-6) + 16(-20) = -80.$
Replacing the first column of the matrix of coefficients with the right–hand side of system (11.3), we get
$\Delta_x = 1\cdot12 - 0(-6) + 0(-20) = 12.$
Therefore, $x = \Delta_x/\Delta = 12/(-80) = -0.15.$
Substituting $x = -0.15$, from the first and the second equation of system (11.3) we obtain

$$\begin{cases} 6y + 4z = 1 - 13(-0.15), \\ 8y + 2z = 0 - 14(-0.15), \end{cases} \Leftrightarrow \begin{cases} 6y + 4z = 2.95, \\ 8y + 2z = 2.1. \end{cases}$$

The determinant of the coefficients matrix for this system is
$\Delta_{(1)} = 6\cdot2 - 8\cdot4 = -20.$
Correspondingly, $\Delta_{(1)y} = 2.95\cdot2 - 2.1\cdot 4 = -2.5$, and therefore, $y = -2.5/(-20) = 0.125.$
Now, using the fact that $y = 0.125$, from the equation $6y+4z=2.95$ we determine that $z=(2.95 - 6\cdot0.125)/4 = 0.55.$

So, we got the 1st column of the inverse matrix: $\begin{pmatrix} -0.15 \\ 0.125 \\ 0.55 \end{pmatrix}$.

To find the 2nd column of the inverse matrix we solve the system

$$\begin{cases} 13x + 6y + 4z = 0, \\ 14x + 8y + 2z = 1, \\ 16x + 6y + 3z = 0. \end{cases} \qquad (11.4)$$

It has the same coefficients matrix as system (11.3).

Therefore, $\Delta = 13\cdot 12 - 14(-6) + 16(-20) = -80$,

$\Delta_x = 0\cdot 12 - 1(-6) + 0(-20) = -6$.

Then, $x = \Delta_x/\Delta = -6/(-80) = -0.075$.

Substituting $x = -0.075$, from the first and the second equation of (4) we obtain

$$\begin{cases} 6y + 4z = 0 - 13(-0.075), \\ 8y + 2z = 1 - 14(-0.075), \end{cases} \Leftrightarrow \begin{cases} 6y + 4z = 0.975, \\ 8y + 2z = 2.05. \end{cases}$$

The determinant of the coefficients matrix for this system is already calculated: $\Delta_{(1)} = -20$.

Correspondingly, $\Delta_{(1)y} = 0.975\cdot 2 - 2.05\cdot 4 = -6.25$, and therefore, $y = -6.25/(-20) = 0.3125$.

Using the fact that $y = 0.3125$, from the equation

$6y + 4z = 0.975$ we determine that

$z = (0.975 - 6\cdot 0.3125)/4 = -0.225$.

Then, the 2nd column of the inverse matrix is $\begin{pmatrix} -0.075 \\ 0.3125 \\ -0.225 \end{pmatrix}$.

Solving the system the same way $\begin{cases} 13x + 6y + 4z = 0, \\ 14x + 8y + 2z = 0, \\ 16x + 6y + 3z = 1, \end{cases}$

we get the 3rd column of the inverse matrix: $\begin{pmatrix} 0.25 \\ -0.375 \\ -0.25 \end{pmatrix}$.

As a result of the calculations, we have a matrix inverse to the coefficients matrix of system (11.2):

$$A^{-1} = \begin{pmatrix} -0.15 & -0.075 & 0.25 \\ 0.125 & 0.3125 & -0.375 \\ 0.55 & -0.225 & -0.25 \end{pmatrix}.$$

If you take the common factor out of the brackets: *1 / (determinant of the matrix A)*
the inverse matrix can be written in a form more convenient for calculations:

$$A^{-1} = \frac{1}{-80}\begin{pmatrix} 12 & 6 & -20 \\ -10 & -25 & 30 \\ -44 & 18 & 20 \end{pmatrix}.$$

Now, as noted earlier, in order to determine the rules by which the points are counted at Bakai's University, it is sufficient to calculate the product $A^{-1}F_1$.
Correspondingly, we determine the rules of scoring at Sanjar's University from product $A^{-1}F_2$; at Samat's from $A^{-1}F_3$; Kyzjibek's University from $A^{-1}F_4$.
Since

$$A^{-1}F_1 = \frac{1}{-80}\begin{pmatrix} 12 & 6 & -20 \\ -10 & -25 & 30 \\ -44 & 18 & 20 \end{pmatrix}\begin{pmatrix} 41 \\ 38 \\ 40 \end{pmatrix} =$$

$$= \frac{1}{-80}\begin{pmatrix} -80 \\ -160 \\ -320 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

it turns out that Bakai's University for each shot from under the basket awards 1 point, for each shot from the middle distance — 2 points, for each shot from the long distance — 4 points.
Doing similar calculations, it is not difficult to answer the remaining questions. To write down a solution in a more compact form, it is convenient to put F_1, F_2, F_3 and F_4 in a single matrix F:

$$F = \begin{pmatrix} 41 & 123 & 83 & 80 \\ 38 & 124 & 84 & 80 \\ 40 & 123 & 87 & 80 \end{pmatrix},$$

and to put the process of determining the rules as the product of $A^{-1}F$.
So,

$$A^{-1}F = \frac{1}{-80}\begin{pmatrix} 12 & 6 & -20 \\ -10 & -25 & 30 \\ -44 & 18 & 20 \end{pmatrix}\begin{pmatrix} 41 & 123 & 83 & 80 \\ 38 & 124 & 84 & 80 \\ 40 & 123 & 87 & 80 \end{pmatrix} =$$

$$= \frac{1}{-80}\begin{pmatrix} -80 & -240 & -240 & -160 \\ -160 & -640 & -320 & -400 \\ -320 & -720 & -400 & -480 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 8 & 4 & 5 \\ 4 & 9 & 5 & 6 \end{pmatrix}.$$

The columns of the matrix $A^{-1}F$ show how the points are calculated in each case. Thus, the fourth column indicates that according to Kyzjibek's rules for each shot from under the basket 2 points are awarded, for each shot from the middle distance — 5 points, for each shot from the long distance — 6 points.

Exercises 11.1.

C. 1) Solve system

a) $\begin{cases} 25x + 37y = 53, \\ 27x + 40y = 57. \end{cases}$ b) $\begin{cases} 25x + 37y = -34, \\ 27x + 40y = -37. \end{cases}$

c) $\begin{cases} 25x + 37y = 171, \\ 27x + 40y = 185. \end{cases}$

2) Sasha patronizes three families. For Tanya's family, she buys 3 kilograms of carrots, 5 kilograms of potatoes and 1 kilogram of rice weekly. For Semen's family: 4 kilograms of carrots and 7 kilograms of potatoes; for Almaz's family: 2 kilograms of carrots and 6 kilograms of rice. Determine at what price food was bought if 244 soms were spent on products for Tanya's family in the first week; for Semen's family — 212 soms; for Almaz's family — 576 soms. What were the prices in the second, third and fourth weeks if the expenses in the corresponding weeks for the Tanya's family amounted to: 240 soms; 267 soms; 235.5 soms; for the Semen's family: 213 soms; 230 soms; 223.5 soms; for the Almaz's family: 550 soms; 638 soms; 522 soms.

H. 1) Find a matrix inverse to given: $A = \begin{pmatrix} 5 & 3 & 2 \\ 7 & 4 & 0 \\ 8 & 5 & 7 \end{pmatrix}.$

2) Students patronize two orphanages. For orphanage A, they buy 23 kilograms of sweets and 35 kilograms of cookies every month, for orphanage B – 34 kilograms of sweets and 50 kilograms of cookies. Determine at what price candies and cookies were bought if 9500 soms were spent in the first week for orphanage A; for orphanage B – 13800 soms. What were the prices in the second, third and fourth weeks if the costs in the respective weeks for orphanage A amounted to: 9260 soms; 9170 soms; 9850 soms; for orphanage B: 13480 soms; 13340 soms; 14300 soms.

11.2. Inverse matrix by the Gauss method

To demonstrate the determining of the inverse matrix by the Gauss method, consider the following scenario:

Problem

Three farmers who rear wheat, corn and fattening bulls, have agreed to follow common politics: to make only certain amount of production and to sell at uniform prices.
The quota for the first farmer was 30 tons of wheat, 50 tons of corn and 40 tons of beef, for the second – 20, 60 and 50 tons, for the third farmer – 60, 20 and 30 tons accordingly.
The cost of the first farmer was c_1*, of the second was* c_2*, of the third farmer was* c_3*.*

Solution

In order to determine the prices at which the costs will be fully covered, it is necessary to solve the system:

$$\begin{cases} 30x + 50y + 40z = c_1, \\ 20x + 60y + 50z = c_2, \\ 60x + 20y + 30z = c_3. \end{cases}$$

Here, x, y, and z are prices per kilogram, c_1, c_2, c_3 are costs in thousands of $ on each farm.
Due to the fact that the prices of the factors of production are changing, costs will change too.

Therefore, in order to be able to quickly get a solution for any values of c_1, c_2, c_3 it is convenient to use an inverse matrix. Finding the inverse matrix A^{-1} by the Gauss method starts with writing down a super–augmented matrix in the form of $(A \mid I)$, where I is the identity matrix. Then, using elementary operations on the rows of the matrix, matrix A is converted into the identity matrix. Then, instead of the identity matrix the inverse matrix of matrix A appears: $(A \mid I) \Rightarrow (I \mid A^{-1})$.

For the coefficients matrix of the system in the problem about farmers super–augmented matrix has the form:

$$\left(\begin{array}{ccc|ccc} 30 & 50 & 40 & 1 & 0 & 0 \\ 20 & 60 & 50 & 0 & 1 & 0 \\ 60 & 20 & 30 & 0 & 0 & 1 \end{array}\right).$$

We transform this matrix. The transformation process will be described by notations used in the Gauss method.

$$\left(\begin{array}{ccc|ccc} 30 & 50 & 40 & 1 & 0 & 0 \\ 20 & 60 & 50 & 0 & 1 & 0 \\ 60 & 20 & 30 & 0 & 0 & 1 \end{array}\right) \frac{R_1 = r_1 - r_2}{}$$

$$\left(\begin{array}{ccc|ccc} 10 & -10 & -10 & 1 & -1 & 0 \\ 20 & 60 & 50 & 0 & 1 & 0 \\ 60 & 20 & 30 & 0 & 0 & 1 \end{array}\right) \frac{R_2 = r_2 - 2r_1}{R_3 = r_3 - 6r_1}$$

$$\left(\begin{array}{ccc|ccc} 10 & -10 & -10 & 1 & -1 & 0 \\ 0 & 80 & 70 & -2 & 3 & 0 \\ 0 & 80 & 90 & -6 & 6 & 1 \end{array}\right) \frac{}{R_3 = r_3 - r_2}$$

$$\left(\begin{array}{ccc|ccc} 10 & -10 & -10 & 1 & -1 & 0 \\ 0 & 80 & 70 & -2 & 3 & 0 \\ 0 & 0 & 20 & -4 & 3 & 1 \end{array}\right) \frac{}{R_3 = r_3 : 20}$$

$$\left(\begin{array}{ccc|ccc} 10 & -10 & -10 & 1 & -1 & 0 \\ 0 & 80 & 70 & -2 & 3 & 0 \\ 0 & 0 & 1 & -0.2 & 0.15 & 0.05 \end{array}\right) \frac{R_1 = r_1 + 10r_3}{R_2 = r_2 - 70r_3}$$

$$\left(\begin{array}{ccc|ccc} 10 & -10 & 0 & -1 & 0.5 & 0.5 \\ 0 & 80 & 0 & 12 & -7.5 & -3.5 \\ 0 & 0 & 1 & -0.2 & 0.15 & 0.05 \end{array}\right) \underline{R_1 = r_1 + r_2 : 8}$$

$$\left(\begin{array}{ccc|ccc} 10 & 0 & 0 & 0.5 & -0.4375 & 0.0625 \\ 0 & 80 & 0 & 12 & -7.5 & -3.5 \\ 0 & 0 & 1 & -0.2 & 0.15 & 0.05 \end{array}\right) \frac{R_1 = r_1 : 1}{R_2 = r_2 : 80}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.05 & -0.04375 & 0.00625 \\ 0 & 1 & 0 & 0.15 & -0.09375 & -0.04375 \\ 0 & 0 & 1 & -0.2 & 0.15 & 0.05 \end{array}\right).$$

The process of finding the inverse matrix is completed. If we factor out the factor *1/160 = 0.00625*, the inverse matrix will be written in a form more convenient for calculations:

$$\begin{pmatrix} 0.05 & -0.04375 & 0.00625 \\ 0.15 & -0.09375 & -0.04375 \\ -0.2 & 0.15 & 0.05 \end{pmatrix} = \frac{1}{160} \begin{pmatrix} 8 & -7 & 1 \\ 24 & -15 & -7 \\ -32 & 24 & 8 \end{pmatrix}.$$

As a result, we have an appropriate tool for the right pricing policy. So, if we know that the costs of farmers are $c_1 = 177.5$; $c_2 = 217$; $c_3 = 139$ thousand dollars, in order to cover their costs, they have to sell their products at the following prices:

$$\frac{1}{160} \begin{pmatrix} 8 & -7 & 1 \\ 24 & -15 & -7 \\ -32 & 24 & 8 \end{pmatrix} \begin{pmatrix} 177.5 \\ 217 \\ 139 \end{pmatrix} =$$

$$= \frac{1}{160} \begin{pmatrix} 8 \cdot 177.5 + (-7) \cdot 21\,7 + 1 \cdot 139 \\ 24 \cdot 177.5 + (-15) \cdot 217 + (-7) \cdot 139 \\ -32 \cdot 177.5 + 24 \cdot 217 + 8 \cdot 139 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.2 \\ 4 \end{pmatrix}$$

So, the price of wheat is \$0.25; corn is \$0.2; meat is \$4 per kilogram.

At the same time, if they plan to cover all expenses and get some profit equal to 31.5; 38; 22 thousand dollars accordingly, the prices should be:

$$\frac{1}{160} \begin{pmatrix} 8 & -7 & 1 \\ 24 & -15 & -7 \\ -32 & 24 & 8 \end{pmatrix} \begin{pmatrix} 177.5 + 315 \\ 217 + 38 \\ 139 + 22 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.4 \\ 4.5 \end{pmatrix}.$$

Exercises 11.2.

C. Find a matrix inverse to given, if it exists:

1. $\begin{pmatrix} -4 & 2 \\ -1 & 0.5 \end{pmatrix}$; 2. $\begin{pmatrix} 1 & -6 \\ 2 & -3 \end{pmatrix}$;

3. $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$ 4. $\begin{pmatrix} 21 & 17 & 31 \\ 23 & -1 & 19 \\ 17 & 53 & 55 \end{pmatrix}$

H. Find a matrix inverse to given, if it exists:

1. $\begin{pmatrix} -14 & 5 \\ 7 & -3 \end{pmatrix}$; 2. $\begin{pmatrix} 33 & -69 \\ -11 & 23 \end{pmatrix}$;

3. $\begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & -4 \\ 1 & 2 & 5 \end{pmatrix}$ 4. $\begin{pmatrix} 11 & 107 & 91 \\ 23 & -1 & 19 \\ 17 & 53 & 55 \end{pmatrix}$

Summary 1

1–10. Find a matrix inverse to given, if it exists.

1. $\begin{pmatrix} 4 & 5 \\ 9 & 7 \end{pmatrix}$; 2. $\begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix}$;

3. $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 3 & 2 \end{pmatrix}$; 4. $\begin{pmatrix} 0 & 3 & 1 \\ 1 & 1 & 0 \\ 2 & 3 & 3 \end{pmatrix}$

5. $\begin{pmatrix} -51 & -29 & 52 \\ 7 & 4 & -7 \\ 9 & 5 & -9 \end{pmatrix}$; 6. $\begin{pmatrix} 2.5 & -0.5 & -0.5 \\ -1 & 1 & 0 \\ -0.5 & -0.5 & 0.5 \end{pmatrix}$

7. Prove, that the matrix $\frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is inverse to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a non-zero determinant Δ.

8. Solve the basketball problem from 11.1 using the Gauss method.

9. Solve the farmer problem from 11.2 using the Cramer-Gauss method.

Appendix . *Cryptography*.

The art of writing and reading encrypted (secret) messages is called cryptography. Not only spies and secret agents need it. Nowadays, in the age of computers, cryptography is very popular due to the rapid development of information technology.
In this section we take a look at the simplest way to create and read secret messages.

11.3. Problem

Once a famous pirate Jack Sparrow was suddenly arrested and jailed. While in prison, he received a package from friends. There were some bread and fish, a bottle with a little rum at the bottom (guards drank the rest), and a deck of cards. After examining the order of the cards in the deck, Jack learned the cause of his arrest. How did he do it?
It turns out that his crew developed a secret code — each card represented a certain letter. Therefore, putting the cards in the correct order, they could send each other messages, which outsiders could not read.
Let's also join the circle of friends of Jack Sparrow. We agree that two of spades stands for letter A, three of spades stands for letter B, and so on, as shown in table 11. 1.

A	B	C	D	E	F
♠2	♠3	♠4	♠5	♠6	♠7
G	H	I	J	K	L
♠8	♠9	♠10	♠Jack	♠Queen	♠King
N	O	P	Q	R	S
♣2	♣3	♣4	♣5	♣6	♣7
T	U	V	W	X	Y
♣8	♣9	♣10	♣Jack	♣Queen	♣King
A	B	C	D	E	F
♦2	♦3	♦4	♦5	♦6	♦7
G	H	I	J	K	L
♦8	♦9	♦10	♦Jack	♦Queen	♦King
N	O	P	Q	R	S
♥2	♥3	♥4	♥5	♥6	♥7
T	U	V	W	X	Y
♥8	♥9	♥10	♥Jack	♥Queen	♥King

Table 11. 1.

Friends of Jack Sparrow gave him a deck, in which the cards were stacked as follows:

♥3 ♣2 ♠6 ♦6 ♣King ♦6 ♠5 ♥8 ♣3 ♦Ace ♠10 ♣7 ♠2 ♥8 ♣6 ♦2 ♦10 ♥8 ♣3 ♣6

To read the message Jack used table 11.1:

♥3	♣2	♠6	♦6	♣King	♦6	♠5	♥8	♣3	♦Ace
O	**N**	**E**	**E**	**Y**	**E**	**D**	**T**	**O**	**M**
♠10	♣7	♠2	♥8	♣6	♦2	♦10	♥8	♣3	♣6
I	**S**	**A**	**T**	**R**	**A**	**I**	**T**	**O**	**R**

and read „ONE–EYED TOM IS A TRAITOR“.

Exercises 11.3

1) Read the message

♠Queen ♥King ♥6 ♠8 ♣King ♥Ace ♣7 ♣8 ♦2 ♣2

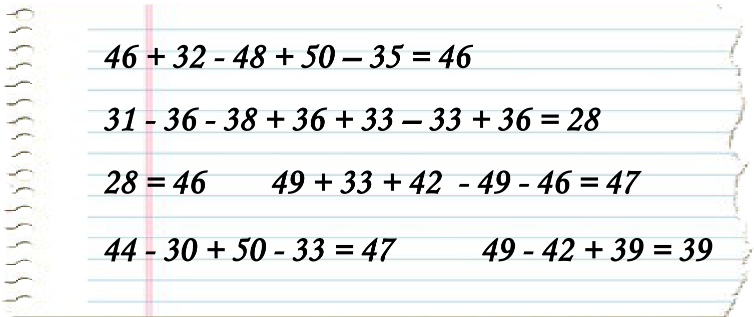
2) Read the message

♣7 ♣3 ♣9 ♥8 ♦9 ♦4 ♠2 ♥4 ♠10 ♥8 ♠2 ♦King

11.4. Problem

After a while, Jack Sparrow received a package with some fish and bread wrapped in a sheet from a school mathematics notebook. A message that was written on this sheet helped Jack escape.

Of course, it was an encrypted message, too. The following was written on the sheet in children's handwriting:



Those who looked through the package did not pay attention to the mathematical exercises, and Jack Sparrow read the note using table 11.2.

A	**B**	**C**	**D**	**E**	**F**	**G**
50	*49*	*48*	*47*	*46*	*45*	*44*
H	**I**	**J**	**K**	**L**	**M**	**H**
43	*42*	*41*	*40*	*39*	*38*	*43*
N	**O**	**P**	**Q**	**R**	**S**	**T**
37	*36*	*35*	*34*	*33*	*32*	*31*
U	**V**	**W**	**X**	**Y**	**Z**	
30	*29*	*28*	*27*	*26*	*25*	

Table 11.2

Let's read it:

46	*32*	*48*	*50*	*35*	*46*				
E	***S***	***C***	***A***	***P***	***E***				
31	*36*	*38*	*36*	*33*	*33*	*36*	*28*		
T	***O***	***M***	***O***	***R***	***R***	***O***	***W***		
28	*46*		*49*	*33*	*42*	*49*	*46*	*47*	
W	***E***		***B***	***R***	***I***	***B***	***E***	***D***	
44	*30*	*50*	*33*	*47*		*49*	*42*	*39*	*39*
G	***U***	***A***	***R***	***D***		***B***	***I***	***L***	***L***

Exercises 11.4 Using table 11.2
1) Read the message
37 – 50 + 33 – 26 = 37
32 – 26 + 33 + 47 – 50 – 33 + 26 = 50
2) Encode the phrase I LOVE MATHEMATICS.
3) Read the message
29 + 50 + 39 – 46 + 33 = 26
48 – 43 + 40 + 50 – 39 + 36 = 29
4) Encode the phrase FRENCH PERFUME

11.5. Problem

To have a more complicated code we will use matrixes.

Example Let's encode a phraze *USEFUL MATH* this way: divide letters into pairs, change each letter into a certain number using table 11.2, and write down each pair as a matrix:

$\binom{U}{S} = \binom{30}{32}$, $\binom{E}{F} = \binom{46}{45}$, $\binom{U}{L} = \binom{30}{39}$,
$\binom{M}{A} = \binom{38}{50}$, $\binom{T}{H} = \binom{31}{43}$.

Then, choose a matrix with nonzero determinant — with an inverse matrix — to transform each column. Let the key matrix be $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. Therefore

$$A \cdot \begin{pmatrix} U & E & U & M & T \\ S & F & L & A & H \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 30 & 46 & 30 & 38 & 31 \\ 32 & 45 & 39 & 50 & 43 \end{pmatrix}$$

$$= \begin{pmatrix} 156 & 227 & 177 & 226 & 191 \\ 94 & 136 & 108 & 138 & 117 \end{pmatrix}.$$

We encode the phraze *USEFUL MATH* as

156 94 227 136 177 108 226 138 191 117.

Exercises 11.5.

1) Using table 11. 2 and key matrix $A = \begin{pmatrix} 3 & 8 \\ 2 & 7 \end{pmatrix}$ encode the word *AMERICAN*.

2) Using table 11.2 and key matrix $A = \begin{pmatrix} 4 & 6 \\ 5 & 7 \end{pmatrix}$ encode the word *SCHOOL*.

11.6. Problem

To decode the message we need to represent each pair of numbers as a column matrix and multiply inverse and message matrices.

Then we replace numbers with letters using table 11. 2. In our case (see **11.5.**) $A^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. Therefore

$$A^{-1} \cdot \begin{pmatrix} 156 & 227 & 177 & 226 & 191 \\ 94 & 136 & 108 & 138 & 117 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 156 & 227 & 177 & 226 & 191 \\ 94 & 136 & 108 & 138 & 117 \end{pmatrix} =$$

$$= \begin{pmatrix} 30 & 46 & 30 & 38 & 31 \\ 32 & 45 & 39 & 50 & 43 \end{pmatrix} = \begin{pmatrix} U & E & U & M & T \\ S & F & L & A & H \end{pmatrix}.$$

Exercises 11.6.

1) Using table 11.2 and key matrix $A = \begin{pmatrix} 3 & 8 \\ 2 & 7 \end{pmatrix}$ read the message *468 382 411 339*.

2) Using table 11.2 and key matrix $A=\begin{pmatrix} 4 & 6 \\ 5 & 7 \end{pmatrix}$ read the message *492 590 316 379*.

11.7. Problem

The code will be more complicated if we split the message into 3 or more letter groups. At the same time it is necessary to take a matrix of the 3rd order or more with a determinant different from zero as the key matrix. If there are not enough letters for the last group, we can agree to complement it with specific letters. It can be the first letters of the message:

$MATHEMATICIAN \Leftrightarrow MAT \quad HEM \quad ATI \quad CIA \quad NMA.$

Example Using a key matrix $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ and

A	**B**	**C**	**D**	**E**	**F**	**G**
1	*2*	*3*	*4*	*5*	*6*	*7*
H	**I**	**J**	**K**	**L**	**M**	**N**
8	*9*	*10*	*11*	*12*	*13*	*14*
O	**P**	**Q**	**R**	**S**	**T**	**U**
15	*16*	*17*	*18*	*19*	*20*	*21*
V	**W**	**X**	**Y**	**Z**		
22	*23*	*24*	*25*	*26*		

Table 11.3

let's read the message *12 15 22 5 9 19 1 12 12*.

We begin the process of decoding with an inverse matrix. As it was stated above, different methods can be used. Let's use the Cramer-Gauss method. We should solve three systems of equations. The left-hand side of each system consists of the key matrix $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ elements as factors of unknowns, while the right-hand side is a column of an identity matrix.

First, we find the first column of the inverse matrix — solve a system $\begin{cases} 2x + 0y + z = 1, \\ x + 3y + 0z = 0, \\ x - y + 2z = 0. \end{cases}$

The determinant of the matrix of system's coefficients is $\Delta = 2 \cdot 6 - 1 \cdot 1 + 1 \cdot (-3) = 8.$

Changing the first column of the matrix of coefficients on the right–hand side of the system, we shall receive, that $\Delta_x = 1 \cdot 6 - 0 \cdot 1 + 0 \cdot (-3) = 6$. Therefore $x = 6/8$.

Substituting the found value in the two first equations of the system, we get $y = -2/8; z = -4/8.$

Further, examining systems with the right-hand sides $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we get the second and the third columns of the inverse matrix. Then

$$A^{-1} = \frac{1}{8}\begin{pmatrix} 6 & -1 & -3 \\ -2 & 3 & 1 \\ -4 & 2 & 6 \end{pmatrix}$$

Now, we have to multiply the inverse matrix by a matrix obtained from the numbers of the encryption and to read the message:

$$\frac{1}{8}\begin{pmatrix} 6 & -1 & -3 \\ -2 & 3 & 1 \\ -4 & 2 & 6 \end{pmatrix}\begin{pmatrix} 46 & 29 & 14 \\ 57 & 32 & 37 \\ 41 & 34 & 13 \end{pmatrix} =$$

$$\begin{pmatrix} 12 - L & 5 - E & 1 - A \\ 15 - O & 9 - I & 12 - L \\ 22 - V & 19 - S & 12 - L \end{pmatrix}.$$

Exercises 11.7.

1) Using a key matrix $\begin{pmatrix} 1 & 2 & 2 \\ 4 & 9 & 8 \\ -1 & -1 & 3 \end{pmatrix}$ and a table 11.3 read the message $55 \quad 221 \quad 46 \quad 84 \quad 345 \quad 45$.

2) Using a key matrix $\begin{pmatrix} -1 & -1 & 4 \\ 1 & 0 & -7 \\ 6 & 7 & -14 \end{pmatrix}$ and a table 11.3 read the message $3 \quad -47 \quad 89 \quad 3 \quad -21 \quad 35$.

Summary 2

1. Encode the message *A SPY IN CITY* using a matrix $\begin{pmatrix} 5 & -2 \\ 3 & 7 \end{pmatrix}$ and the table 11.3.
2. Encode the message *WE SHALL BREAK* using a matrix $\begin{pmatrix} 7 & -8 & 3 \\ 4 & 5 & -5 \\ -3 & 0 & 8 \end{pmatrix}$ and the table 11.3.
3. The message $-58 \; 130 \; -89 \; 241 \; -282 \; 466$ was encoded using the matrix $\begin{pmatrix} 11 & -27 \\ 5 & 19 \end{pmatrix}$ and the table 11.3. Decode this message.
4. The message $419 \; 497 \; 489 \; 582 \; 469 \; 572$ was encoded using the matrix $\begin{pmatrix} 17 & 19 \\ 21 & 22 \end{pmatrix}$ and the table 11.3. Decode this message.
5. The message $-10 \; -1 \; -10 \; 44 \; 44 \; -8$ was encoded using the matrix $\begin{pmatrix} -2 & -1 & 11 \\ -1 & 0 & 7 \\ -1 & -1 & 3 \end{pmatrix}$ and the table 11.3. Decode this message.

6. The message $0\ 14\ 5\ -20\ \ 47\ \ -5$ was encoded using the matrix $\begin{pmatrix} 3 & -3 & 2 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}$ and the table 11.3. Decode this message.

7. The message $50\ \ 29\ -20\ \ 69\ \ 63\ \ 8$ was encoded using the matrix $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{pmatrix}$ and the table 11.3. Decode this message.

§12. Introduction in linear programming. The geometric approach

In the Soviet encyclopedic dictionary for 1984 there are only 4 words which are devoted for the term of linear programming: «one of the sections of mathematical programming». It can be accident, but to some extent, it expresses the relation of the Soviet science, in particular mathematics, to applied aspects.

Linear programming allows, using a simple mathematical apparatus, to solve very important and interesting practical problems. This feature makes linear programming one of the key points in the training of the future economists, businesspersons, etc.

It is necessary to notice that the major contribution to development of the theory of linear programming has been made by Soviet scientist L. D. Kantorovich who was noted for this work by the Nobel Prize on economy (1975). But as often happens, these results in Soviet Union practically were not used, but were developed and actively applied in the West.

Expression of a kind $a_1x_1 + a_2x_2 + ... + a_nx_n$, where a_1 , a_2 , a_n – constant numbers, is called linear. The problem on a finding of the maximum (minimum) value of linear function in the area defined by system of the linear equations and inequalities, is called as a problem of linear programming.

Problem

The shop has 5 kg of chocolates and 8kg of caramel and sells these sweets in packages of two kinds. Package A contains 100gr chocolates and 400gr caramel. Package B contains 250 g of sweets of both kinds. How many packages of each kind the shop needs in order to maximize profit if package A gives \$7 and package B gives \$4 of profit?

Solution

Having denoted quantity of package A through *x* and quantity of package B through *y*, we have an opportunity to write this problem in the mathematical language:

$$\begin{cases} x \geq 0; \quad y \geq 0; \\ 0.4x + 0.25y \leq 8; \\ 0.1x + 0.25y \leq 5; \\ Pf = 7x + 4y \rightarrow max. \end{cases}$$

The first inequalities are obvious – the quantity of packages cannot be negative; second inequality is defined by restriction on chocolates, third – on caramel.

The problem 1 can be solved in two ways: geometric and algebraic.

The geometric approach to the solution of problems of linear programming is pretty simple and descriptive. However, this approach suits only to the solution of problems with two variables. When there are more than two variables, it is necessary to use an algebraic way.

12.1.The geometric approach to the solution of problems of linear programming

With the sufficient basis on that, it is possible to assert that the point set by the equation $x = 5$ divides a numerical axis OX on two rays. The objections, consisting that $x = 5$ is not that point and the point $x = 0$ divides the axis OX into half-rays, are possible. We will accept the compromise conclusion — both, a point $x = 5$ and a point $x = 0$ divide the axis OX into half-rays. The base for this conclusion is the fact that in both cases points divide the axis OX into parts with the length equals ∞.

Each half-ray is defined by the inequality: for instance $x \geq 5$ in the sense that the coordinate of each point of a belonging half-ray satisfies to the inequality. Similarly, we say that each straight line set by the equation $Ax + By = C$ divides a plane into semi-planes; each semi-plane is set determined by an inequality of kind $Ax + By \geq C$ or $Ax + By \leq C$.

Example

To define a semi-plane which is set by an inequality $2x + 3y \leq 12$, it is necessary to draw a straight line $2x + 3y = 12$. So as a result two semi-planes will turn out: one of them is set by an inequality $2x + 3y \leq 12$, another, by $2x + 3y \geq 12$.

To choose the necessary semi-plane, it is enough to take one certain point M and to substitute its coordinates in an initial inequality. If we receive a true inequality the point M lies in defined semi space; if not, the opposite semi space is defined. As a point of M we recommend to take a point $O(0, 0)$ if the free member C is distinct from zero and a point on one of the axes of coordinates, if $C = 0$.

Let's return to our example. We will substitute coordinates of point $O(0, 0)$ in the inequality $2x + 3y \leq 12$ and we will receive a true inequality $2 \cdot 0 + 3 \cdot 0 \leq 12$. Hence, it is necessary to choose from two semi-planes that in which the point *O* is lying. On a drawing, we recommend to delete (to shade) the semi-plane which is not chosen.

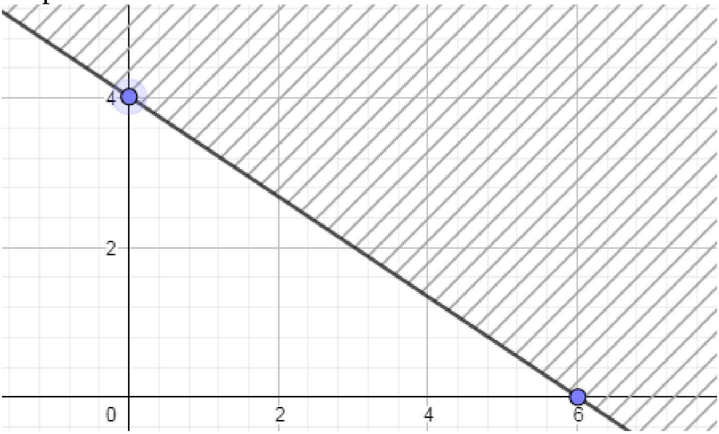


Figure 12.1

12.2. A problem on a maximum

Let solve a problem 1 by the geometric approach:

$$\begin{cases} x \geq 0; \quad y \geq 0; \\ 0.4x + 0.25y \leq 8; \\ 0.1x + 0.25y \leq 5; \\ Pf = 7x + 4y \to max. \end{cases} \qquad (12.1)$$

In graphical language of an inequality $x \geq 0$ and $y \geq 0$ it is said that it is necessary to consider only the first quarter.

The restriction on chocolates $0.1x + 0.25y \leq 5$ will be represented as:

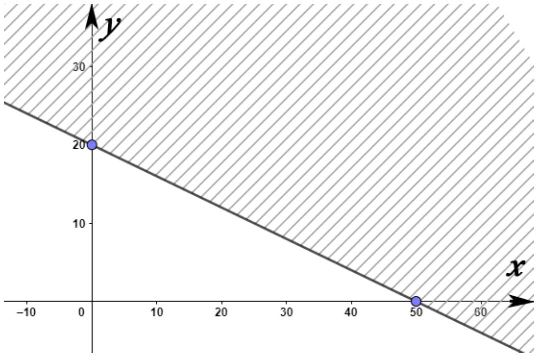


Figure 12.2

And restriction on caramel $0.4x + 0.25y \leq 8$ as follows:

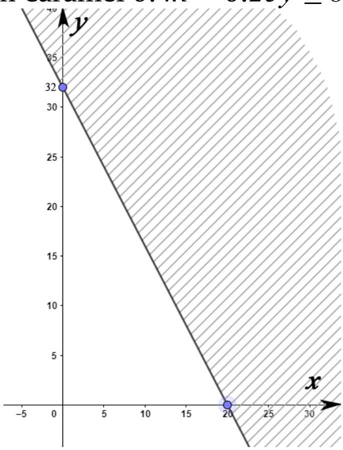


Figure 12.3

Let's unite graphs: so we will receive quadrangle *KLMO*.

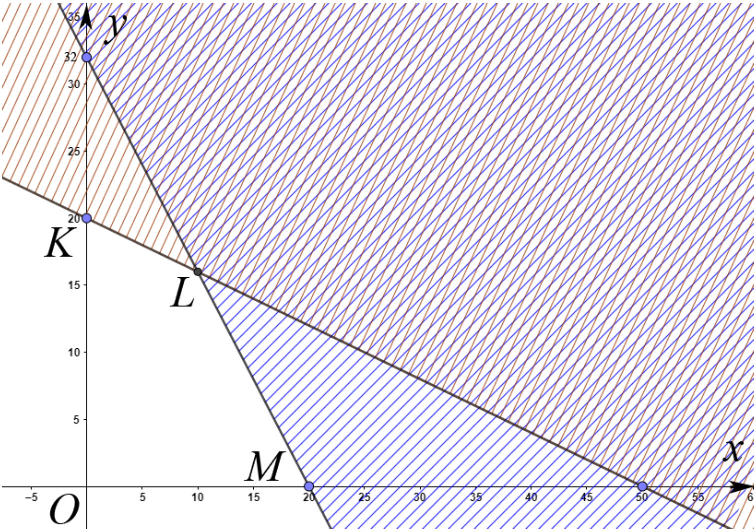


Figure 12.4

Each point of this quadrangle (its coordinates) shows sets of *A* and *B* which the shop can have. For example, a point with coordinates (*10, 10*) belongs to *KLMO*. It means that having 5 kg of chocolates and 8 kg of caramel, the shop can prepare 10 packing's of both *A* and *B*. At the same time the point *(10, 20)* does not lie on *KLMO*. It means that it is impossible to prepare 10 packing's of *A* and 20 of *B* simultaneously. Then, it is reasonable to denote *KLMO* feasible region.
Further it will be important to know coordinates of point *L*. For their identification, we will solve a system of the form:

$$\begin{cases} 0.1x + 0.25y = 5, \\ 0.4x + 0.25y = 8, \end{cases} \Rightarrow \begin{cases} x = 10, \\ y = 16. \end{cases}$$

So, the shop has the ability to produce a considerable quantity of various sets from packages of kinds *A* and *B*. Further, we will speak, about the ways of finding a set that will provide the greatest profit.

Common profit is set by the function: $Pf = 7x + 4y$ which is called as objective function. Having equated Pf to zero, we receive a straight line $7x + 4y = 0$, passing through the origin of coordinates. With growth of value *Pf* the straight line will be displaced parallel to the right side, upwards. For example, when *Pf = 84* we have a straight line passing through the points *S(0, 21)* and *T(12, 0)*.

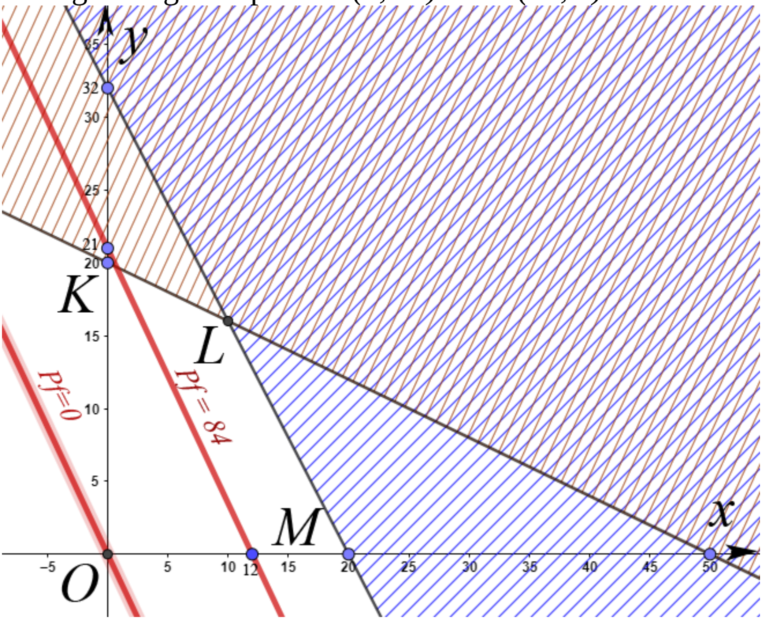


Figure 12.5

It is obvious that the feasible area contains points that are lying more to the right and upwards then line *ST*, that means they have sets that provide higher profits. Continuing in the same spirit, we come to the basic theorem of the theory of linear programming:

Theorem *Objective function reaches a maximum (minimum) in one of vertexes of the polygon, which is the feasible area.*

The theorem proof can be found in the special literature devoted to the linear programming.

From the previous theorem we can conclude that the shop will get the maximum profit, having made and having sold a set corresponding to one of the following points: *K(0, 20)*, *L(10, 16)*, *M(20, 0)*. To be defined, it is enough to calculate values of objective function in these points:

$Pf(K) = Pf(0,20) = 7·0 + 4·20 = 80;$

$Pf(L) = Pf(10,16) = 7·10 + 4·16 = 134;$

$Pf(M) = Pf(20,0) = 7·20 + 4·0 = 140.$

Hence, the shop can get a maximum profit, equal to 140, having made and having sold 20 sets of the kind *A*.

Problem

If, with other things in the problem from 12.1 being equal, the profit from one packing of a kind *A* makes $5 instead of $7, objective function $Pf = 5x + 4y$ will reach maximum in point *L*, because

$Pf(K) = Pf(0,20) = 5·0 + 4·20 = 80;$

$Pf(L) = Pf(10,16) = 5·10 + 4·16 = 114;$

$Pf(M) = Pf(20,0) = 5·20 + 4·0 = 100.$

It means that to maximize profit the shop should sell 10 sets of kind *A* and 16 sets of kind *B*.

Exercises 12.1

C. A jewelry firm can produce two kinds of bracelets from 672 g of gold and 864 g of silver, spending 2700 working hours. Bracelet A requires 8g of gold, 9 g of silver and 30 working hours; bracelet B requires 12 g, 18 g and 60 hours accordingly. How to maximize the profit, if it is known, that bracelet A gives $320 of profit and bracelet B gives $576?

H. A firm produces two types of water skies, a trick ski and a slalom ski. The relevant manufacturing data of one unit of skies are given in the table.

Department	Labor-Hour per Ski		Maximum Labor–Hour Available per Day
	Trick Ski	Slalom Ski	
Fabricating	6	4	108
Finishing	1.2	1	24

What is the maximum profit, if the profit on a trick ski is $35 and the profit on a slalom ski is $28?

12.3. A problem on a minimum

Problem *For improvement of quality of diesel fuel various additives are used. In particular, each ton of fuel of mark "Alpha" should contain not less than 40 mg of additive X, not less than 14 mg of the additive Y, and not less than 18 mg of the additive Z. These additives are contained in products A and B. Content of additives in each liter of a product is resulted in the table (in mg):*

	X	***Y***	***Z***
A	*4*	*2*	*3*
B	*5*	*1*	*1*

It is required to find a set of products A and B, that minimize cost of additives if 1 liter of a product A costs 50 cents, 1 liter of a product B costs 60 cents.

Solution

Let's rewrite a problem in mathematical language, having denoted through *a* and *b* liters

of products A and B:
$$\begin{cases} a \geq 0, \quad b \geq 0, \\ 4a + 5b \geq 40, \\ 2a + b \geq 14, \\ 3a + b \geq 18, \\ C = 50a + 60b \rightarrow min \end{cases}$$

The feasible area of a problem on the schedule looks as follows.

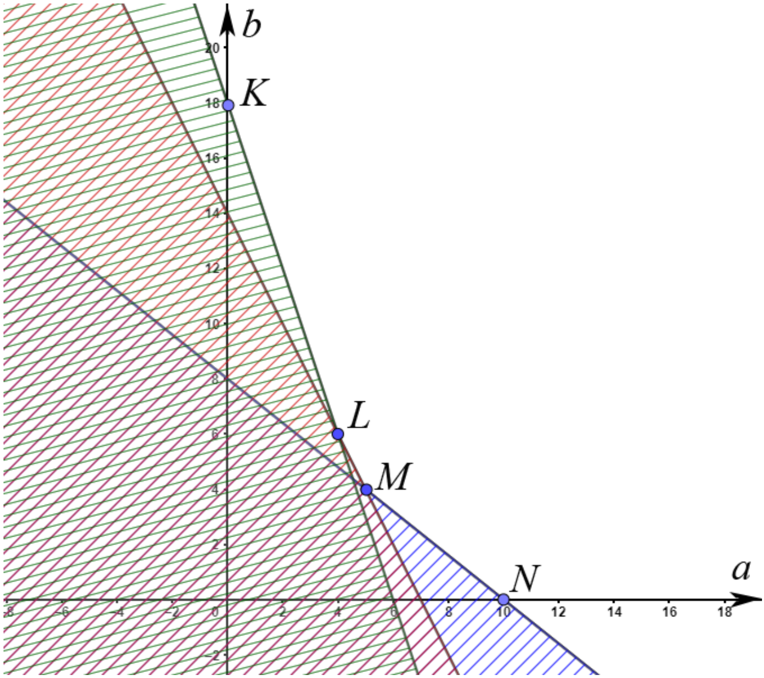


Figure 12.6

Let's notice that to construct straight lines limiting feasible area, it is convenient to use writing in the form of the equation in segments. ($4a + 5b = 40$ rewrite as $\frac{a}{10} + \frac{b}{8} = 1$ and so on). From the theorem (see the previous chapter) follows that objective function $C = 50a + 60b$ reaches the minimum in one of the angular points: *K*, *L*, *M*, *N*.

From constructions follows that the point *K* has coordinates (*0*, *18*), point *N* (*10*, *0*). For finding of coordinates of points L and M we will solve the systems of the equations which describe straight lines, on the crossing of which these points are lying.

$$L: \begin{cases} 2a + b = 14, \\ 3a + b = 18 \end{cases} \Rightarrow \begin{cases} a = 4, \\ b = 6. \end{cases}$$

$$M: \begin{cases} 4a + 5b = 40, \\ 2a + b = 14 \end{cases} \Rightarrow \begin{cases} 3b = 12, \\ 2a + b = 14 \end{cases} \Rightarrow \begin{cases} a = 5, \\ b = 4. \end{cases}$$

For the completion of the solving process of a problem it is necessary to calculate values of a objective function in angular points.

$C(K) = 50 \cdot 0 + 60 \cdot 18 = 1080;$ $C(L) = 50 \cdot 4 + 60 \cdot 6 = 560;$
$C(M) = 50 \cdot 5 + 60 \cdot 4 = 490;$ $C(N) = 50 \cdot 10 + 60 \cdot 0 = 500.$

So, in order to receive a ton of fuel of mark "Alpha" it is enough to spend 490 cents, having used 5 liters of a product *A* and 4 liters of a product *B*.

Except the answer to the question which we get from the conditions of the problem, by analyzing solving process, it is possible to receive a lot of other helpful information. In particular, the optimum point – the *M* point is defined by requirements to additives *X* and *Y*. This means that under additive *Z* there is a reserve: $3 \cdot 5 + 4 = 19$. That is, under requirements to mark "Alpha" it is enough to have 18 units of additive *Z*, and we are going to add 19 units.

Exercises 12.2.

C. To improve the quality of the motor oil, in one ton of oil must be added not less than 165 units of substance H and 110 units of substance M. For this purpose can be used mixture X, which contains 10 units of H, 5 units of M, and costs 10 soms, and mixture Y with corresponding values: 11, 11 and 17 soms. How to improve the quality of the oil and spend minimum money?

H. A patient in a hospital is required to have at least 84 units of drug A, 120 units of drug B and 100 units of drug D each day. Each portion of substance M contains 10 units of drug A, 8 units of drug B and 5 units of drug D, and each portion of substance N contains 2 units of drug A, 4 units of drug B and 5 units of drug D. What minimum of money can meet the requirement, if cost of the portion M is $0.7, of the portion N is $0.3?

12.4. A transport problem.

Problem *A firm has 400 boxes with goods in the 1st storehouse and 500 in the 2nd. 350 of them are necessary to deliver to the shop A, 300 to the shop B. How many boxes should be delivered from each storehouse in each shop in order to minimize transport expenses if delivery of one box from storehouse 1 to the shop A costs \$2.5; from 1 to B costs \$1.5; from 2 to A costs \$2; from 2 to B costs \$1.8?*

Solution

It will be useful to rewrite data in the form of a table:

	Shop A	Shop B	Reserve
1st storehouse	*2.5*	*1.5*	*400*
2nd storehouse	*2*	*1.8*	*500*
Requirements	*350*	*300*	

Let's denote x as the quantity of boxes delivered from 1st storehouse to shop A; y is the quantity of boxes delivered from 1st storehouse to shop B. So,

$(350 - x)$ is the quantity of boxes delivered from 2nd storehouse to shop A;

$(300 - y)$ is the quantity of boxes delivered from 2nd storehouse to shop B.

As a result we have the problem at the mathematical language:

$$\begin{cases} x \geq 0; \quad y \geq 0; \\ 350 - x \geq 0; \\ 300 - y \geq 0; \\ x + y \leq 400; \\ (350 - x) + (300 - y) \leq 500; \\ C = 2.5x + 1.5y + 2(350 - x) + 1.8(300 - y) \to min. \end{cases}$$

Having resulted in like terms, we will obtain following:

$$\begin{cases} x \geq 0; \quad y \geq 0; \\ x \leq 350; \\ y \leq 300; \\ x + y \leq 400; \\ x + y \geq 150; \\ C = 0.5x - 0.3y + 1240 \to min. \end{cases}$$

From the kind of function, *C* follows that for the achievement of minimum it is necessary to have a point with a small value of *X* and big value of *Y*. We will draw a feasible area:

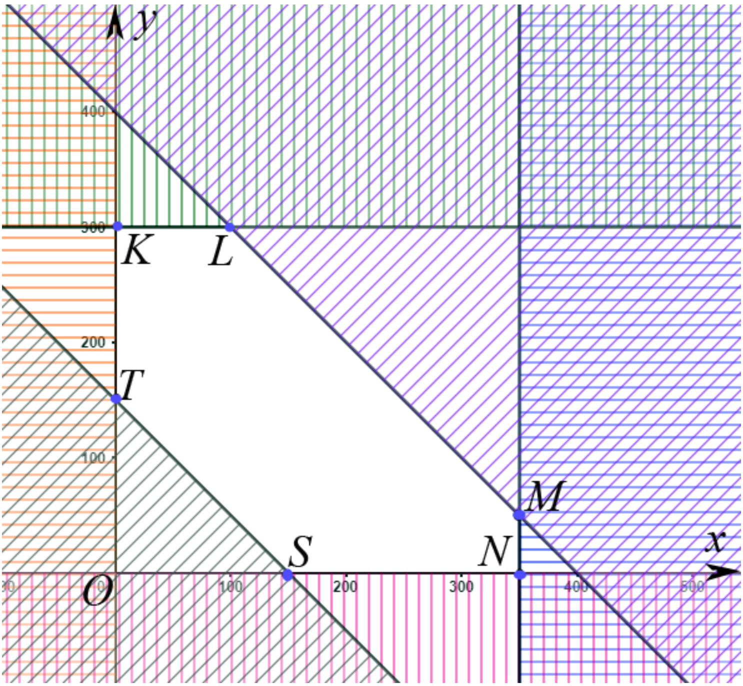


Figure 12.7

In the feasible area, polygon *KLMNST*, the smallest value *X* and greatest value *Y* has the point *K* is point (*0, 300*).

Hence, the minimum of transport costs is equal to:

$C(K) = 0.5 \cdot 0 - 0.3 \cdot 300 + 1240 = \$1150.$

It will be reached, if transportation is transported out under the following plan:

	Shop A	Shop B
1 storehouse	*0*	*300*
2 storehouse	*350*	*0*

Exercises 12.3.

C. From warehouses K and L to the store A it must be delivered 120 TV and 70 TV to the store B. The warehouse K contains 140 TV and warehouse L contains 90 TV. Shipping costs are: $8 per TV to ship from K to A; $6 per TV to ship from K to B; $7 per TV to ship from L to A; $9 per TV to ship from L to B. What are the minimum transport costs?

H. From warehouses K and L to the store A it must be delivered 320 bags, to the store B is 210. What are the minimum transport costs if each warehouse contains 280 bags and shipping costs for one bag are given in the table below?

	A	B
K	*2*	*3*
L	*1*	*1.6*

12.5. Closed transport problem

Problem

Let's add conditions to the problem 4: 250 boxes which have remained in storehouses should be delivered to the shop C. How the delivery plan of the shop will be looked, if it is known that a delivery of one box to the shop C from the storehouse 1 costs $3, from the storehouse 2 costs $2.4?

The table of transport expenses will be as follows

	Shop A	Shop B	Shop C	Reserves
1 storehouse	*2.5*	*1.5*	*3*	*400*
2 storehouse	*2*	*1.8*	*2.4*	*500*
Requirements	*350*	*300*	*250*	

Solution

Feature of this problem is that the volume of reserves: $400 + 500 = 900$ coincides with the volume of requirements: $350 + 300 + 250$.

Therefore, despite the occurrence of the shop C, there is no necessity to enter a new variable. Such problems are called as ***closed***.

The quantity of boxes delivered from the storehouse 1 to the shop C will be denoted through $400 - x - y$, from the storehouse 2 to the shop C will be denoted through

$500 - (350 - x) - (300 - y) = x + y - 150$.

Restrictions corresponding to the closed transport problems consist of the requirement of non-negativity of quantity of the goods transported from each storehouse to each shop:

$$\begin{cases} x \geq 0; \quad y \geq 0; \\ 350 - x \geq 0; \\ 300 - y \geq 0; \\ 400 - x - y \geq 0; \\ x + y - 150 \geq 0; \\ C = 2.5x + 1.5y + 3(400 - x - y) + 2(350 - x) + \\ \quad + 1.8(300 - y) + 2.4(x + y - 150) \to min. \end{cases}$$

The function of costs after reduction of like terms:

$C = -0.1x - 0.9y + 2080$.

Pay attention — the feasible area has not changed.

The theory says that the minimum will be reached in one of the angular points; objective function coefficients dictate that it is necessary to choose points with the greatest values x and y, that is, it is point L or M.

Values of the objective function in these points:

$C(L) = -0.1 \cdot 100 - 0.9 \cdot 300 + 2080 = \$1800$

C(M) = –0.1·350 – 0.9·50 + 2080 = $2000

Answer Transporting according to the plan:

	Shop A	Shop B	Shop C
1 storehouse	*100*	*300*	*0*
2 storehouse	*250*	*0*	*250*

the firm will have the minimum transport expenses equal to $1800.

Exercises 12.4.

C. From bread factories A and B delivered 8 tons of bread to settlement K, 12 to L, 12 to M. How to minimize transportation costs, if the factory A has 16 tons; B has 16 tons, and the cost of transportation of one ton of bread are in the table given below:

	K	**L**	**M**
A	*19.5*	*18*	*19*
B	*18*	*19*	*18*

H. From bakeries A and B 110 ton of bread is transported to the city K, 120 ton to the N, 150 ton to the M. How can we organize the delivery of bread to cities, so that the total cost of transportation will be minimal, if bakery A has 120 ton, B has 260 ton, and the transportation cost for each ton of bread are drawn in the table below:

	K	**L**	**M**
A	*11*	*12*	*10*
B	*9*	*11*	*8*

Summary

1. A firm makes two kinds of porridge. The basic restrictions imposed on output – presence of monthly fund of working hours in each of three shops of factory: 1000 hours per shop A, 360 hours per shop B, 600 per shop C. Number of man-hours demanded for manufacture for 1centner of a product are set by the table:

	Shop A	Shop B	Shop C
1 porridge	*10*	*3*	*2*
2 porridge	*4*	*2*	*4*

How many centners of each porridge should be produced in order to maximize profit if 1 centner of the first porridge brings \$500 of profit, 1 centner of the second porridge brings \$600?

1a. Solve problem 1, changing 600 hours per shop C by 1200 hours per shop C; \$600 brings 1 centner of the second porridge by \$300.

2. A firm makes two types of TVs. The TV of the 1st type demands 6 hours on assemblage, 3 hours on adjustment and 2 hours for checking. The TV of the 2nd type, accordingly: 5; 5; 1h. How many TVs of each type should be produced in order to get maximum profit if the TV of the 1st type brings \$40 of profit, 2nd – \$50? It is known that at assemblage it is possible to use 900 hours, at adjustment – 600 hours, at check – 280 hours, and under the contract with the shop it is necessary to produce not less than 20 TVs of the 1st type and 40 TVs of the 2nd type.

3. The firm SAN produces two kinds of TVs. Each TV of mark X demands 6 hours on assemblage and 1 hour on packing and brings \$90 of profit. Each TV of mark Y demands 10 hours on assemblage and 1 hour on packing and brings \$160 of profit. How many TVs of each kind SAN firm should produce in order to maximize profit if the firm can use 780 hours on assemblage and 110 on packing?

After a while SAN firm, besides assemblage and packing, had a necessity to make adjustment. For adjustment of the TV of mark X 2 hour is required, for the TV of mark Y 5 hours is required, and in total there are 340 hours. How many TVs of each kind SAN firm should produce in order to maximize profit?

At the third stage it was found out that the firm can sell no more than 39 TVs of mark Y. What is the greatest possible profit level in this case is equal?

4. In a forage for chickens there is a daily requirement, so it should contain not less than 2600 units of vitamin A and 2000 units of vitamin D. Food additive AY contains 30 units of vitamin A and 20 units of vitamin D and costs $5. The corresponding data under additive SL: 20; 40; $3. How to meet requirements, having spent a minimum of money?

5. For the improvement of quality of diesel some chemicals are added. In a ton of diesel fuel not less than 40 mg additives X should be added, not less 14 mg of additive Y and not less than 18 mg Z. These additives contain in products A and B. Content of additives in each liter of a product is resulted in the table (in mg):

	X	**Y**	**Z**
A	*4*	*2*	*3*
B	*5*	*1*	*2*

Find a set of products A and B that minimize the cost of additives, if:

a) 1 liter of product A costs $50, 1 liter of product B costs $20;

b) 1 liter of product A costs $45, 1 liter of product B costs $30.

6. A firm produces soft drinks at two factories: R and T. From R and T 4000 bottles should be delivered to shop A, and 3500 to shop B. Factory R can deliver not more than 7500, and factory T – 4000 bottles. The information on cost of transportation of one bottle from each factory to each shop is resulted in the table (in cents):

	A	B
R	*4*	*2.5*
T	*3*	*2*

How can we organize the delivery of bottles to shops so that the total cost of transportation will be minimum?

6a. A firm produces soft drinks at two factories: R and T. From R and T 4000 bottles should be delivered to shop A, and 3500 to shop B. Factory R can deliver not more than 4500, and factory T – 4000 bottles. The information on cost of transportation of one bottle from each factory to each shop is resulted in the table (in cents):

	A	B
P	*2.5*	*2.5*
T	*3*	*2*

How can we organize the delivery of bottles to shops, so that the total cost of transportation will be minimum?

7. How can the delivery be organized if in statements of the problem 6 there will be shop C, which needs 4000 bottles, cost of delivery from factory R is equal to 2 cents,
from T — 4 cents?

7a. How can the delivery be organized if in statements of the problem 6a there will be shop C, which needs 1000 bottles, cost of delivery from factory R is equal to 2 cents, from factory T – 4 cents?

8. From storehouses R and S there should be the delivery of 200 bags to shop A, to shop B – 500. What is the minimum of transport costs if in storehouse R there are 550 bags, in storehouse S – 400 bags, and expenses on delivery of one bag are set by the table:

	A	B
R	*2*	*3*
S	*1*	*1.5*

9. From storehouses K and L there should be the delivery of 500 bags to shop A, to B – 100, to C – 200. What is the minimum of transport costs if in each storehouse 400 bags are, and expenses on delivery of one bag are set by the table:

	A	B	C
K	20	17	10
L	23	12	12

10. What is the maximum and minimum value of the function $z = 5x + 2y$ subject to:

$x \geq 0,\ y \geq 0,\ 3y - 2x \geq 0,\quad y + 8x \leq 52,\quad y - 2x \leq 2,\quad x \geq 3.$

11. Find the value $x \geq 0$ and $y \geq 0$ which minimize the function $C = 3x + 2y$ subject to:

a) $10x + 7y \leq 42,\quad 4x + 10y \geq 35.$

b) $6x + 5y \geq 25,\quad 2x + 6y \geq 15.$

c) $x + 2y \geq 10,\quad 2x + y \geq 12,\quad x - y \leq 8.$

12. Maximize $z = 7x + 9y$ subject to: $x \geq 0,\ y \geq 0,$ $y + 2x \leq 10,\quad 3y + 5x \leq 26.$

13. Minimize $z = 3x + 6y$ subject to: $x \geq 0,\ y \geq 0,$ $x + y \leq 20,\ 3x + y \geq 15,\ x + 2y \geq 15$.

14. Maximize and minimize $z = 5x + 8y$ subject to:

$x \geq 1, y \geq 2, x+2y \leq 20, 3x+y \leq 15, x+y \geq 7.$

15. Maximize $z = 6x + 4y$ subject to: $x \geq 0;\ y \geq 0;$ $1.2x + 0.6y \leq 960;\ 0.04x + 0.03y \leq 36;\ 0.2x + 0.3y \leq 270.$

Summary for chapters 1–12

1. Little Red Riding Hood may get to Grandma across the three paths. First, with the length *a* meters, leads to the House of woodcutters, second, with the length *b* meters, leads from the home of woodcutters to Wolf's shelter, and the third path, with the length *c* meters, runs from the shelter of the Wolf to grandmother's House. When little Red Riding Hood asked about the length of paths, her mother, who teaches math, said that if from 2*a* subtract 4*b*, and add *c* to the result, than it will be the 100 meters. At the same time, if subtract *a* from 8*b* and add 2*c*, you get 3000 meters, and if 3*a* add 5*b* and 6*c*, you get a 5700 meters. When little Red Riding Hood came and told the grandmother about it, she said that the Chief Forester Èl′Aman likes to repair paths. So, a month ago, instead of numbers 100, 3000 and 5700 would number 700, 2100 and 5500, and two months ago, 200, 2700 and 5400. Help Red Riding Hood determine the length of the paths in each month.

2. The costs of a firm at produce 18 units of goods was 2140 soms, at produce 26 units is 2890 soms. The revenue was 2460.7 soms, when 22 units of goods was sold. Assume that the linear dependence takes place and find the break–even point.

3. The company makes two types of cake. Biking cake of the first type takes 40 minutes, second type is 24 minutes; creamy covering of the first type takes 15 minutes, second type is 18 minutes. How to maximize profits, if there are 80 hours for the baking, 45 hours for creamy covering, and you cannot sell more than 120 units of the second kind of cake, a cake of the first kind is worth $2 profit, the second $2.5?

4. In daily dose of feed for chickens it must be contained at least 120 units of vitamin A, 52 units of vitamin B and 84 units of vitamin D. Dietary supplement K contains 8 units of vitamin A, 3 units of vitamin Band 10 units of vitamin D and cost $2.2. Corresponding data for the dietary supplement S: 4; 2; 2 and $1. How to meet the requirements and spend a minimum of money?

5. From A and B milk factories 4; 6 and 5 tons of milk is transported to K, L and M cities accordingly. How can we organize the delivery of milk to cities, so that the total cost of transportations will be minimal, if A factory produces 6.5 tons of milk, B is 8.5 tons, and transportation fees for one ton of milk are drawn in the table below:

	K	L	M
A	*120*	*140*	*110*
B	*140*	*155*	*130*

6. From warehouses K and L to the store A it must be delivered 220 bags, to the store B is 310.What are the minimum transport costs if each warehouse contains 300 bags and shipping costs on one bag are given in table below?

	A	B
K	*22*	*30*
L	*10*	*12*

7. Decode the message

128 –14 65 118 –12 61 22 –37 2

which was encoded by conformity

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
14	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*

and the matrix $\begin{pmatrix} 3 & 5 & -2 \\ -2 & 1 & 0 \\ 1 & 3 & -1 \end{pmatrix}$

8. Firm A sold 30% of itself production in Bishkek, 50% – in Almaty and 20% – in Tokmok. Similar data about firm B: 50%, 30%, 20%; firm C: 35%, 20%, 45%. How many units of goods was manufactures on the each firm if Bishkek consumers received 94 units, Almaty consumers received 78 units, Tokmok consumers received 68.

9. The costs of preparation for the issuance and sale of a new type of product of $13960. Draw the graph and determine the profit zone, knowing that the average variable cost for the first 500 units are $9.1 for subsequent $7.5, while a price (in $) is a)*41.1*; b) *28*; c) *64 – 0.05q*.

10. The points *A*(*– 1, 4), B(– 4, 1), C(5, – 2*), *D(7, 5*). Find:

a) the angle between vectors *AB* and *AC*;

b) the angle between vectors *AB* and *AD*;

c) the square of the quadrilateral *ABCD;*

d) a coordinates of points *K* — a vertex of a parallelogram *ABCK*;

e) the equation of a straight line *CL*, which is a parallel to the straight line *AB*;

f) the equation of a straight line *DP*, perpendicular to the straight line *BC*.

11. Investigate the system on solvability and write solutions for each particular case:

$$\begin{cases} 5x + (7 - a)y = 10, \\ ax + 2y = 4. \end{cases}$$

Answers and directions

Chapter 1

Exercises 1.1. C. *{37; 43};* **H.** *{73; 213}*;
Exercises 1.2. C. {*53; 18* }; **H.** *493* cm^2;
Exercises 1.3. C. *152* cm; *171* cm; **H.** *90.5* soms;
Exercises 1.4. C.*126* **H.** *{240; 140};*
Exercises 1.5. C.*126* **H.** *{240; 140}***;**
Exercises 1.6. C.*340* **H.** *{520; 730};*
Exercises 1.4. C. Karim's weight is 71 kg, Dania's weight is 54 kg, Tariel's weight is 68 kg.
H. 996g, 1003g, 1006g, 998g. Average weight is 1000.75g.
Exercises 1.8. C. {*47; 79; – 26*} **H.**{*63; 37*}.
Summary
1. {*– 2; 4*} **2.** {*2/17; 7/17*} **3.** {*20; 64*} **4.** {*1.5; 0,8*}
5. {*6.5; 9*} **6.** {*7; 12*} 7. The price of the notepad is $1.8, the album is $1.5.
8. The price of the pen is $0.28, the ruler is $0.35 and the notebook is $0.37.

Chapter 2

Exercises 2.1
C. $\overline{EF} = (5;4)$, $\overline{FE} = (–5;–4)$, $\overline{EG} = (3;–3)$, $\overline{FG} = (–2;–7)$.
H. $\overline{KL} = (5;5)$, $\overline{LM} = (1;–7)$, $\overline{ML} = (–1;7)$, $\overline{KM} = (6;–2)$.
Exercises 2.2
C. 1) *a)* $\overline{EG} = \overline{EF} + \overline{FG}$; *b)* $\overline{FI} = \overline{FG} + \overline{GH} + \overline{HI}$;
c) $\overline{GF} = -\overline{FG}$; *d)* $\overline{EI} = \overline{EF} + \overline{FG} + \overline{GH} + \overline{HI}$.
2) $\overline{EF} = (5;4)$, $\overline{FG} = (–2;–7)$, $\overline{GH} = (–1;0)$, $\overline{HI} = (–1;1)$
a) $\overline{EG} = (3–3)$; *b)* $\overline{FI} = (–4;–6)$ *c)* $\overline{GF} = (2;7)$; *d)* $\overline{EI} = (1;–2)$.

H. 1) *a)* $\overline{NM} = -\overline{MN}$; *b)* $\overline{LN} = \overline{LM} + \overline{MN}$;
c) $\overline{LO} = \overline{LM} + \overline{MN} + \overline{NO}$; *d)* $\overline{KO} = \overline{KL} + \overline{LM} + \overline{MN} + \overline{NO}$;
2) $\overline{KL}$ *= (5; 1),* $\overline{LM}$ *=(2; –6),* $\overline{MN}$ *=(–2; 1),* $\overline{NO}$ *=(–5; 7)*
a) $\overline{NM}$ *=(2; –1);* *b)* $\overline{LN}$ *=(0; –5)*
c) $\overline{LO}$ *=(–5;2)* *d)* $\overline{KO}$ *= (0;3)*

Exercises 2.3

C. 1) *T=(–11.2;50/7);* 2) *T = (1; –2)+(–3;–5) = (–2;–7)*
H. *1) P=(–11;14); 2)D=(3; –5)+(–2;–5) = (1;0)*

Exercises 2.4

C. $\sqrt{1^2+(-3)^2} = \sqrt{10} \approx 3.16$
H. $\sqrt{10^2+(1)^2} = \sqrt{101} \approx 10.04$

Exercises 2.5

C. $\sqrt{24^2+(-24)^2} = \sqrt{1152} \approx 33.94$
H. $\sqrt{6^2+(-4)^2} = \sqrt{52} \approx 7.21$

Exercises 2.6

C. *(7/25;–24/25)* **H**. *(6/10;8/10)*

Exercises 2.7

C. *(–90/41;400/41), (90/41;–400/41).*
H. *(–20/13; 48/13), (20/13; –48/13).*

Exercises 2.8

C. *R (–1; 5.5);* **H**. *C (4;–1)*

Exercises 2.9

C. 68 m; **H**. 65 m.

Exercises 2.10

C. *600* **H**. *1140*

Exercises 2.11

C. *–30;* **H**. *–6.5*

Exercises 2.12

C. *arccos* $(11/\sqrt{17\cdot 45})$

H.The angle between the vectors $\overline{AB}$ and $2019\bar{a}$ equals the angle between the vectors $\overline{AB}$ and $\bar{a}$ *:arccos(13/$\sqrt{65\cdot 26}$)*

Exercises 2.13

C. *(17/6; 1/3)* **H**. *(–1/22; –37/22)*

Exercises 2.14

C. $1.3\bar{a} + 2\bar{b} = 5.6i - 0.1j - 7.5k;$ *–101;*

H. $-0.4\bar{a} + 6\bar{b} = 21.6i + 5.6j;$ *–22.2;*

Summary

1. a) $\overline{AC}$ **;** b) $\overline{BA}$ **;** c) $\overline{AE}$ **;** d) $\overline{AB}$ **;**

2. a) $\sqrt{180}$ **;** b) $\sqrt{160}$ **;**

3. a) *7/3*; b) $x = 15$; $y = 1.2$;

4. a) *AB*(*2; 2*); $\sqrt{8}$ and *AC*(*0; 6*); 6; b) (*6; 3*);
c) (*8; 5*) or (*0; –3*); d) (*8/3; –1/3*) ; e) 24;

5. a) (*0; 4*); b) (*–0.2; 0.6*) ; c) (*9; –12*) or (*–9; 12*);
d) *arccos (26/$\sqrt{1000}$);*

6. *1140;*

7. a) (*15/*$\sqrt{3.25}$; *–10/*$\sqrt{3.25}$);
or (*–15/*$\sqrt{3.25}$;*10/*$\sqrt{3.25}$) **;** b) (*24/7; –6/7*) ;
c) *arccos (2/*$\sqrt{13\cdot 65}$ *) ;* d) $\sqrt{82}$ **;** 5) *14.5*;
e) (*–98/29; –54/29*);

8. $-4\bar{a} + 1.6\bar{b} = 25.6i - 15.2j$; *– 13;*

Chapter 3

Exercises 3.1

C. *a)16; b)40; c)3; d)3.75;*

H. *a)15; b)35; c)4 ; d)2.8;*

Exercises 3.2

C. *1. a)26; b)50; c)13; d)13.75;*
2. a)11; b)35; c) – 2; d) – 2.75;

H. *1. a)17; b)37; c)6; d)4.8; 2. a)23; b)43; c)12; d)10.8;*

Exercises 3.3

C. $p_S = 4q - 20$; $p_D = -5/4q + 127$;

H. $p_S = 62.5q + 25$; $p_D = -40q + 221.8$.

Exercises 3.4

C. *a)140; b)260; c)2.5; d)1.25;*

H. *a)60; b)20; c)7; d)10;*

Exercises 3.5

C. $a(w) = 45w + 70$; 358;

H. $v(t) = -8t + 94$; *26;*

Exercises 3.6

C. a) $y = 4x - 2$; b) $y = -0.2x + 4$;

H. a) $y = -2x + 3$; b) $y = 3x - 3$.

Exercises 3.7

C. $p_S = 4q - 30$; $p_D = -5/4q + 117$; *E(28;82)*

H. $p_S = 62.5q + 35$; $p_D = -40q + 231.8$; *E(1.92;155).*

Exercises 3.8

C. 1:50 PM. Note: starting point for Mukhtar *(90; F)*,the endpoint for Dinara *(140; F).*

H. 72 km; at 5:30 PM.

Summary

1. 28 shirts; 7 days.

2. $q = -1.6t + 9.6$; *2.8* tons; 2:12 PM.

3. a) $y = -x$; b) $y = 0.3x + 4$**.**

4. $p_s = 1.25q - 16.25$; $p_d = -1.25q + 112$; 47.875soms; 51.3 tons.

5. $p_s = 125q + 67$; $p_d = -40q + 331$; *267 soms; 1.6 tons.*

6. 60.4 km. Note. The function $y = -56x + b$ takes place.

7. The bus and taxi met exactly in the middle of the road at 4.8 hours, that is, at 4:48 PM.

8. They met at 4:15PM at a distance of 10 km from Lukashovka.

9. At 2:07 PM and 30 seconds.

Chapter 4

Exercises 4.1
C. a) 5000 soms; b) 6000 soms; c) 8000 soms;
H. a) $2000; b) $2600; c) $2800.
Exercises 4.2
C. a) 3200 soms; b) 5600 soms; c) 7200 soms;
H. a) $900; b) $1200; c) $1800.
Exercises 4.3
C. a) 17800 soms; b) 26600 soms; c) 29000 soms;
H. a) $2100; b) $2400; c) $2900.
Exercises 4.4
C. a) 1 000 soms ; b) 6 500 soms ; c) 10 000 soms.
If Aidana had 20500 soms, then the line should be drawn in parallel, increasing all monetary values by 1500.
H. a) $900; b) $690 ; c) $130.
If Nilufar has $1 450, then the line must be drawn in parallel, reducing all monetary values by 80.
Exercises 4.5
C. a) –1200 soms; b) 200 soms; c) 1000 soms;
H. a) – $200; b) – $100; c) $300.
Exercises 4.6
C. 125 T-shirts; **H**. 650 kilograms of apricots.
Exercises 4.7
C. a) 140; b) 165; c) 134;
H. a) 672 kg; b) 722 kg; c) 660 kg.
Exercises 4.8
C. 172 soms; **H**. $2.3.
Exercises 4.9
C. Aidana should choose the second option if she expects to sell at least 17 units and the first option with lower sales volumes.
H. Salim must choose the second option if he expects to rent bicycles for at least 14 hours and the first option with fewer hours.

Exercises 4.10
C. Aileen should choose the second option if she expects to sell at least 30 and no more than 55 hot dogs and the first option for other sales volumes.
H. Said should choose the second option if he expects to sell at least 1 425 and no more than 1 740 copies of the book and the first option for other volumes of sale.
Exercises 4.11
C. [2008; ∞); **H.** [530; ∞).
Exercises 4.12
C. [425; 680]; **H.** [420; 550].
Exercises 4.13
C. [520; ∞); **H.** [387.5; ∞).
Exercises 4.14
C. [550; ∞); **H.** [400; ∞).
Exercises 4.15
C. [650; ∞); **H.** [500; ∞).
Exercises 4.16
C. [450; 600]; **H.** [420; 550].
Exercises 4.17
C. [450; 600]; **H.** [420; 550].
Exercises 4.18
C. [275; ∞); **H.** [400; ∞).
Exercises 4.19
C. [417.5; ∞); **H.** [555; ∞).
Exercises 4.20
C. [3000; 5300]; **H.** [50; 85].
Summary
1a. [12.5; +∞)**; 1b.** [10; 20]; **2a. [**10; +∞); **2b. [**5; 14]; **3a. [**10; +∞); **3b. [**10; 30]**; 4a. [**40; +∞); **4b. [**12; 50]; **5a. [**40; 135]; **5b. [**30; 45]; **6a. [**50; +∞); **6b. [**50; 65] **;**
7. 615; **8.** *Euro = –1.0144· $ + 96.65858.* Therefore, *–1.0144·43.5151 + 96.65858 = 52.5169.*

9. 21 hour and 3 min; **10.** 25120; **11.** [30; 70]; **11a.** [40; 60]; **11b.** [40; 52.5]; **12a.** **[**250; +∞); **12b**. [405; +∞); **13.** [300; 500]; **14.** [133; +∞); **15.** **[**100; 180]; **15a.** **[**120; 160]; **15b.** [120; 150]; **16a.** [835; +∞); **16b**. [1060; +∞); **17.** [800; 1000]; **18.** **[**100/3; 100].

19a. If the writer believes that it will be sold more than 12 000 copies of the book, then he should work with the firm Q, otherwise the firm P.

19b. If the writer believes that it will be sold more than 210 000 copies of the book, then he should work with the firm Q, otherwise the firm P.

20. If Meerim believes that it will be sold between 150 and 283 it is necessary to work for the firm K, otherwise for M.

21. [340; 625]; **22.** **[77**5; +∞); **23.** [450; 600]; **24.** [400; +∞); **25.** [56; 80].

Chapter 5

Exercises 5.1

C. (7; –6), (52; –51), (–48; 49), (–23; 24), (102; –101).
H. (–19; –8), (21; 7), (77; 28), (–83; 32), (1; –0.5)

Exercises 5.2

C. $(x-5)/(-2) = (y+11)/11$; **H.** $(x-1)/(-3) = (y-7)/2$.

Exercises 5.3

C. $(x-11)/7 = (y+2)/(-33)$; **H.** $(x+3)/11 = (y-15)/3$.

Exercises 5.4

C. $y-27 = 12(x+9)$; **H.** $y-2.1 = 1.5(x-1.1)$

Exercises 5.5

C. $18 000 . **H.** 9000 soms.

Exercises 5.6

C. $(x-39/11)/7 = (y-34/11)/(-7)$.
H. $(x-6)/3 = (y-29/7)/4$.

Exercises 5.7

C. $4n + 10p = 200$; $5n + 10p = 200$.
H. $25c + 10t = 250$; $25c + 10t = 300$.

Exercises 5.8

C. *(–5; –1; 21), (3; 0; –5), (–1; –0.5; 8), (75; 9; –239),*
H. *(12; 51; –1), (33; –10; 43), (117;21;219), (–9;112; –45).*

Exercises 5.9

C. $(x-7)/(-3) = (y-12)/(-15) = (z-4)/(-5)$.
H. $(x-17)/(-18) = (y+2)/(-5) = (z-3)/1 = (t-18)/(-7)$.

Exercises 5.10

C. $(x-21)/(-3) = (y+7)/(-15) = (z-1.5)/(-5)$.
H. $(x-12)/(-18) = (y-1.7)/(-5) = (z+5)/1 = (t-6)/(-7)$.

Summary

1. a) $y = -x-2$; b) $y = -x+5$; c) $y = x+1$;
d) *(–2; 0), (–1; –1), (0; –2), (1; –3), (2; –4)*;
e) *K(–1; –1), slope = 4/3;*
f) $\sqrt{111.25}$; g) $x = 5$ or $y = -7$.

2. a) $\dfrac{x+3}{8} = \dfrac{y-4}{-3}$; b) $\dfrac{x-4}{8} = \dfrac{y+3}{-3}$; c) *(–4; 0)*;
d) $\dfrac{x+3}{7.5} = \dfrac{y-4}{-5}$; e) $\dfrac{x+3}{-4} = \dfrac{y-4}{1}$; f) *35*; g) *35.*

3. a) $-3(x-2) + 7(y-4) = 0$; b) $7(x-2) = -3(y-4)$.

4. a) $(x+1)/6 = (y-2)/0$;
b) $(x+2)/6 = (y+3)/8 = (z-1)/5$;
c) $(x-1)/(-8) = (y+1)/3 = (z-2)/(-11) = t/1$.

5. *These lines are identical.*

6. $(x-7)/(-2) = (y+2)/3 = (z-15)/6$.

7. $R = 15.9375t + 3623$; $R(12) = 3814.25$.

Chapter 6

Exercises 6.1

C. 1) *no solution;* 2) $x = 4.6$;
3) $x = p$, where p is any number;
4) $(6.4 + 0.6p - 1.4q;\ p;\ q)$, where p and q are any numbers;
H. 1) *no solution;* 2) $x = -36$;
3) $x = p$, where p is any number;
4) $(0.75 - 2.75p;\ p)$, where p is any number.

Exercises 6.2

C. 1) $x = 29 - 5a$; 2) $x = (3c - 7)/(c + 9)$, *if* $c \neq -9$; *no solution, if* $c = -9$;

3) $x = -(2 + 3d + 12e)/6d$, *if* $d \neq 0$;
no solution, if $d = 0$ *and* $e \neq -1/6$;
infinity many solutions, if $d = 0$ *and* $e = -1/6$.

H. 1) $x = (28 + 3a)/12$;

2) $x = 12/(5c + 2)$, *if* $c \neq -0.4$; *no solution, if* $c = -0.4$;

3) $x = (56e - 21 + 42d + 12e)/(7 + e)$, *if* $e \neq -7$; *no solution, if* $e = -7$ *and* $d \neq -413/42$;
infinity many solutions, if $e = -7$ *and* $d = -413/42$.

Exercises 6.3.

C. 1) ($5a - 34.8$; $-2.1a - 2.1$);

2) ($[3b^2 + 2b - 56]/[b^2 + 12b + 35]$; $-5b/[b + 7]$), if $b \neq -5$ and $b \neq -7$; *no solution, if* $b = -5$ or $b = -7$;

3) ($1.25d/c$; $1.5d$), if $c \neq 0$;
infinity many solutions (p; 0), where p is any number, if $c = 0$; $d = 0$; *no solution, if* $c = 0$; $d \neq 0$.

H. 1) ($2a$; $-a/3$);

2) (3; $-2b/[b + 3]$), if $b \neq -3$; *no solution, if* $b = -3$;

3) ($30 + 4.5d/[4c + 2]$; $0.3d + 2.4$), if $c \neq -0.5$;
infinity many solutions (p; 5), where p is any number, if $c = -0.5$; $d = -26/3$; *no solution, if* $c = -0.5$; $d \neq -26/3$.

Exercises 6.4.

C. 1) ($0.36a + 3.48$; $-0.04a + 0.28$);

2) ($[12.04 - 9b]/[7 - 5b]$; $7/[7 - 5b]$), if $b \neq 1.4$;
no solution, if $b = 1.4$

3) ($[cd + 3c - 3d - 37]/[5c - 37]$; $[11d - 37/[5c - 37]$),
if $c \neq 37/5$; *infinity many solutions (p; 2.5p – 35/11), where p is any number, if* $c = 37/5$; $d = 37/11$;
no solution, if $c = 37/5$; $d \neq 37/11$.

H. 1) 1) ($0.04a + 0.84$; $0.16a - 3.64$);

2) ($[4a + 9]/[5a + 24]$; $17/[5a + 24]$), if $a \neq -8.8$;

no solution, if $a = -8.8$;

3) $([c^2 + 8c - 4]/[-6 + 5c - c^2]; \ [5c + 14]/[-6 + 5c - c^2])$,
if $c \neq 2$ *and* $c \neq 3$; *no solution, if* $c = 2$ *or* $c = 3$.

Exercises 6.5.

C. $(4; \ 5.5)$; **H**. $(2; \ 7.8)$.

Exercises 6.6.

C. No solution ; **H**. No solution.

Exercises 6.7.

C. $(p; \ 15 - 2.5p)$, *where* p *is any number.*

H. $(1.75p + 7; \ p)$, *where* p *is any number.*

Exercises 6.8.

C. 1) Infinity number of solutions; 2) No solutions;
3) (17; 5.3).

H. 1) (15; –2.2) 2) No solutions; 3) Infinity number of solutions.

Summary

1. a) 3.5; b) $5/8$; c) 10.5; d) $-5/16$; e) $5a$; e) -6.5.

2. a) $1.2a$; b) $-0.6d$; c) $-1/c$, *when* $c \neq 0$, *no solution when* $c = 0$; d) $3d/c$ *when* $c \neq 0$, *no solution when* $c = 0, d \neq 0$, *any number when* $c = d = 0$; e) $(5n+2d)/4c$ *when* $c \neq 0$, *no solution when* $c = 0.5$; $n + 2d \neq 0$, *any number when* $4c = 5n+2d = 0$; f) $-7a/26$;
g) $(7a + b - 15)/(6 + a)$ *when* $a \neq -6$,
no solution when $a = -6, b \neq 57$,
any number when $a = -6, b = 57$.

3. a) $(p; \ 0.4p - 1.6)$; b) $(p; \ p - 4)$; c) $(p; \ 2p)$;
d) $(p; \ q; \ 2.6 - 0.4p + 0.6q)$; e) $(p; \ q; \ \{2p - 38q + 14\}/3)$;
f) $(7 - 2p - 3q + 4r; \ p; \ q; \ r)$;
g) $(n - 2p_2 - 3p_3 - \ldots - np_n ; \ p_2 ; \ p_3 ; \ldots \ np_n)$.

4. 22; **5**. profit \$80; costs \$60; **6**. 36 **7**. $(50; \ 40)$;

8. $(280; \ 320)$; **9**. 80 ; **10**. 672.

11. a) $(4.3a - 0.2; \ -2.5a)$; b) If $b = 6$ — no solution;
if $b \neq 6$ there is unique solution
$((76b - 11b^2 + 7)/(6 - b); \ (10b + 7)/(6 - b))$;

c) If $c = 0$ and $d \neq -2.5$ — no solution;
if $c = 0$ and $d = -2.5$ infinitely many solutions
$(p; (p + 40)/17)$, where p is any number;
if $c \neq 0$ there is unique solution
$((35 + 14d)/4c; (35 + 14d + 280c + 48cd)/68c)$.
12. If a is not equal 5.6, then
$(76/(28 + 5a); 3a - 44/(28 + 5a))$;
if $a = 5.6$, then there is no solution.
13. If $b = 6$, then $(p; (4 - 2p)/3)$, where p is any number;
if b is not equal 6, then $(2 - 0.5b; 2)$.
14. If $c = 5$, then there is no solution;
if $c = 3$, then $(p; 1.4 - 0.6p)$, where p is any number;
if c is not equal 5 and 3, then $(7/(c - 5); -7/(c - 5))$.
15. If $d = 6$, then $(p; (4 - 8p)/6)$, where p is any number;
$d = 4$ there is no solution,
for other values of d the system has solution
$(2/(4-d); 4/(d- 4))$.
16. a) $(7; 0)$; b) $(-1.2; 9.8)$; c) no solution;
d) $(p; 0.375p + 4.5)$, where p is any number.
17. a) $p \neq 5$; b) $p \neq 1.6$. **18**. a) $p = 5$; b) $p = -8/11$.
19. a) $p = 15$; b) $p = -8$.
20. a) Infinity number of solutions; b) No solutions;
c) (26; 0).

Chapter 7

Exercises 7.1. **C.** $\{-7; 7\}$ **H.** $\{-7; 7\}$
Exercises 7.3. **C.** *No solution* **H**. *No solution*
Exercises 7.4. **C.** $\{1; -3; 2\}$ **H**. $\{50000; 70000; 80000\}$
Exercises 7.5. **C.** $\{p; -4.5; 7; -44.5 - 5p\}$, *where p is any number*
H. $\{p; 1 - 4p/7; 5p/7 - 1\}$, *where p is any number*
Summary
1. $\{3; 2; -1\}$ **2**. $\{2; -3; 1\}$ **3**. $\{1; 1; 0\}$ 4. $\varnothing$ **5**. $\varnothing$
6. $\{1000; 2200; 1800\}$ **7**. *$3.5*

Appendix

1. $\{5+29b-17a; -2-17b+10a; a; b\}$; **2**. $\varnothing$;

3. $\{5; 0.5; 0; 2.5\}$; **4**. $\{25-4a; 16-2a; a; -4; 0\}$.

Summary for chapters 1–7

1. *a)(a+b)/2; b) (b – a)/2.*

2. a) 30 square units; b) (16,3); c) (–0.8;1.5);

d) $y=-1/3x+10$;

e) $y=0.5x+2.5$. **3**. a) $V = -1400t+10\,000$; b) $V(2)=7\,200$.

4. a) $P_D=-0.03q+500$; b) $P_s= 0.05q+100$; c) (5000;350).

5. a) (200;+∞); b) (741;+∞); c)(70;760). **6.** (75;60).

7. a) *infinity many solutions (p, 1.5p –2.5), where p is any number*, if $a= 10$; no solutions, if $a\neq 10$;

b) infinity many solutions (p, 0.5p –0.75), where p is any number, if $m= -3$; no solutions, if $m= 3$;

$(3/(m-3), -3/(m-3))$, if $m\neq 3$ and $m\neq -3$.

Chapter 8

Exercises 8.1.	**C.** *108*	**H.** *–123*
Exercises 8.2.	**C.** *–848*	**H.** *1440*
Exercises 8.3.	**C**. *(3; 6; –3)*	**H**. *(2; –1; 4)*
Exercises 8.4.	**C.** *1*	**H.** *2*

Exercises 8.5.

C. The eigenvectors correspond to eigenvalue *0* are vectors $\begin{pmatrix} 5a \\ -2a \\ a \end{pmatrix}$; to eigenvalue *4* are vectors $\begin{pmatrix} 3b \\ 2b \\ b \end{pmatrix}$; to eigenvalue *(–1)* are vectors $\begin{pmatrix} 3c \\ -3c \\ c \end{pmatrix}$, where a, b, c are arbitrary numbers.

H. The eigenvectors correspond to eigenvalue $k = 3$ are vectors $\begin{pmatrix} b \\ 4b \end{pmatrix}$; to eigenvalue $k = 5$ are vectors $\begin{pmatrix} a \\ 2a \end{pmatrix}$, where a, b are arbitrary numbers.

Summary **1**. *–32* **2**. *–6* **3**. $2x - x^2 - 3$
4. *0 (2nd column is equal to doubled 1–st column)*
5. *0* **6**. *–173* **7**. $2a^3$ **8**. *24* **9**. *4* **10**. *24* **11**. *(2; –1; 3)*
12. *(1; –1; 1)*
13. *(2194/758; 1264/758; 860/758)* **14**. *(14.25; 6.75; –3)*
15. *(140; 75; 110)* **16**. *(220; 340; 186)* **17.1**. *28* **17.2**. *9*
18. The eigenvectors correspond to eigenvalue *9* are vectors $\begin{pmatrix} b \\ 1.2b \end{pmatrix}$; to eigenvalue *(–2)* are vectors $\begin{pmatrix} a \\ -a \end{pmatrix}$, where a, b are arbitrary numbers.
19. The eigenvectors correspond to eigenvalue *0* are vectors $\begin{pmatrix} -3a \\ a \\ 4a \end{pmatrix}$; to eigenvalue *1* are vectors $\begin{pmatrix} 5b \\ 0 \\ -7b \end{pmatrix}$; to eigenvalue *2* are vectors $\begin{pmatrix} 3c \\ c \\ -4c \end{pmatrix}$, where a, b, c are arbitrary numbers.

Chapter 9

Exercises 9.1. **C.** *(14; 7; 15)* **H.** *(11; 3; –7)*
Exercises 9.2. **C.** *(285; 194)* **H.** *(47000; 33000)*
Exercises 9.3. **C.** *(40; 100; 80)* **H.** *(3000; 200; 150)*
Exercises 9.4. **C.** Problem has no solution. It means there is some mistake in the problems data. **H.** No solution
Exercises 9.5. **C.** (p; $(19p - 28)/13$; $(46 - 34p)/39$), where p — any number.
H. (p; $(821p - 482)/583$; $(24 - 40p)/53$), where p is any number.
Exercises 9.6. **C.** $a = 1$, then infinity number of solutions; $a = -1$, then there is no solution;

if a is not equal -1 or 1, then unique solution.
H. $a = -2$, then there is no solution;
$a = 2$, then infinity number of solutions;
if a is not equal 2 or -2, then unique solution.
Exercises 9.7. **C.** $y = 3x^2 + 10x + 3$ **H.** $y = 4x^2 - 3x - 1$

Summary

1. *150; 225; 180;* **2**. *50; 20; 18 ;* **3**. *45; 50; 70;*
4. *300 kg;* **5**. *(60; 90; 45). Hint:* $\Delta = 0.912$;
6. *(80; 150; 70). Hint:* $\Delta = 261$; **7**. *no solution ;*
8. *(2; –1; 3);* **9**. *(a; 16.5 – 2.5a; 0.5a – 2.5);* **10**.*(2; –155; 5);*
11. *(3; 50; –8)* **12.** $a = 0$ or $a = 2$ or -2, then infinity number of solutions; for other values of a the system has unique solution. **13**. $y = -3x^2 + 2x + 1$

Chapter 10

Exercises 10.1. **C.** $S + T = \begin{pmatrix} 17 & 23.1 & 7 \\ 16 & -6 & 25 \\ 13 & 20.3 & -6 \end{pmatrix}$;

$S + R$ — *impossible;* $T - R$ — *impossible;*

$T - S = \begin{pmatrix} -1 & -18.9 & -23 \\ -28 & 8 & 3 \\ -1 & 15.7 & 2 \end{pmatrix}$; $0.2R = \begin{pmatrix} 0.24 & -0.4 \\ 0.4 & 1 \\ -3.4 & 0.2 \end{pmatrix}$;

$-7T = \begin{pmatrix} -56 & -14.7 & 56 \\ 42 & -7 & -98 \\ -42 & -126 & 14 \end{pmatrix}$; $3S - 5T = \begin{pmatrix} -13 & 52.5 & 85 \\ 96 & -26 & -37 \\ -9 & -11.1 & -2 \end{pmatrix}$.

H. $K + L$ — *impossible;* $L + M$ — *impossible;*

$K + M = \begin{pmatrix} 23 & -4.7 \\ 34 & -10 \\ 0 & 11 \end{pmatrix}$; $L - M$ — *impossible;*

$K - M = \begin{pmatrix} 19 & 0.7 \\ -30 & 0 \\ -3.4 & 9 \end{pmatrix}$; $4M = \begin{pmatrix} 8 & -10.8 \\ 128 & -20 \\ 6.8 & 4 \end{pmatrix}$;

$1.1L = \begin{pmatrix} 8.8 & 23.1 & -8.8 \\ -6.6 & 1.1 & 14 \\ 6.6 & 19.8 & -2.2 \end{pmatrix}$; $3K - 2.5M = \begin{pmatrix} 58 & -0.75 \\ -74 & -2.5 \\ -9.35 & 27.5 \end{pmatrix}$.

Exercises 10.2.

C. $S \cdot T = \begin{pmatrix} 36 & 309.9 & 192 \\ 284 & 237.2 & -104 \\ 18.2 & -55 & -15.8 \end{pmatrix}$; $S \cdot R = \begin{pmatrix} -105 & 102 \\ 63 & -68 \\ 156.6 & -6.5 \end{pmatrix}$;

$T \cdot S = \begin{pmatrix} 62.6 & 158.13 & 175.1 \\ 66 & -100.8 & -135 \\ 436 & -4.6 & 296 \end{pmatrix}$; $R \cdot T$ — *impossible*.

H. $K \cdot L$ — *impossible;* $L \cdot M = \begin{pmatrix} 69.6 & -40.1 \\ 43.8 & 25.2 \\ 584.6 & -108.2 \end{pmatrix}$;

$K \cdot M$ — *impossible;* $L \cdot K = \begin{pmatrix} 185.8 & -106.5 \\ -144.8 & 147 \\ 165.4 & -122 \end{pmatrix}$.

Exercises 10.4. **C.** $S^T = \begin{pmatrix} 9 & 22 & 7 \\ 21 & -7 & 2.3 \\ 15 & 11 & -4 \end{pmatrix}$;

$R^T = \begin{pmatrix} 12 & 2 & -17 \\ -2 & 5 & 1 \end{pmatrix}$; $P \cdot R^T$ — *impossible;*

$10P + S^T = \begin{pmatrix} 89 & 42 & -73 \\ -39 & 3 & 42.3 \\ 75 & 29 & -24 \end{pmatrix}$; $R^T \cdot P = \begin{pmatrix} -18 & -4.6 & -54 \\ -40 & 2.8 & 34 \end{pmatrix}$.

H. $K^T = \begin{pmatrix} 21 & 2 & -1.7 \\ -2 & -5 & 10 \end{pmatrix}$; $L^T = \begin{pmatrix} 8 & -6 & 6 \\ 2.1 & 1 & 18 \\ -8 & 14 & -2 \end{pmatrix}$;

$L^T \cdot M = \begin{pmatrix} -165.8 & 14.4 \\ 66.8 & 7.33 \\ 428.6 & -50.4 \end{pmatrix}$; $.K^T - 2M$ — *impossible;*

$K^T - 2M^T = \begin{pmatrix} 17 & -62 & -5.1 \\ -7.4 & -15 & 8 \end{pmatrix}$.

Exercises 10.5.

C. $\begin{pmatrix} 45.28 & 53 \\ 63.12 & 75.2 \\ 102 & 119 \end{pmatrix}$ **H.** $\begin{pmatrix} 103.5 & 34.5 \\ -26.5 & 23.25 \end{pmatrix}$.

Exercises 10.6. **C.** *(7; 5; 4.5)* **H.** *(20; 18; 15)*

Summary

1. a) $\begin{pmatrix} -0.9 & 10 \\ -0.2 & 2 \end{pmatrix}$, b) $\begin{pmatrix} 13.2 & -5 \\ 0.6 & -16 \end{pmatrix}$, c) *impossible,*

d) $\begin{pmatrix} -6.3 & -25 \\ -0.42 & -9 \end{pmatrix}$, e) $\begin{pmatrix} -7.3 & 30.5 \\ 0.4 & -8 \end{pmatrix}$, f) $\begin{pmatrix} 18 \\ 238/15 \end{pmatrix}$,

g) *impossible,* *h*) $\begin{pmatrix} 9.7 \\ -10 \end{pmatrix}$, *i*) $\begin{pmatrix} 9.7 \\ -10 \end{pmatrix}$, *j*) $\begin{pmatrix} -3 & -0.2 \\ 5 & 4 \end{pmatrix}$,

k) $\begin{pmatrix} 1/3 & 2 \end{pmatrix}$.

3. *a)* $\begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix}$, *b)* $\begin{pmatrix} 3 & -6 \end{pmatrix}$; *c) impossible;* *d)* $\begin{pmatrix} 3 & -12 \end{pmatrix}$;

5. $\begin{pmatrix} 23 & 19 & 10 \\ 12 & 25 & 16 \end{pmatrix} \begin{pmatrix} 24 \\ 48 \\ 60 \end{pmatrix} = \begin{pmatrix} 2064 \\ 2448 \end{pmatrix}$.

6. *1–st semiannual* = $\begin{pmatrix} 28.4 & 68 \\ 30.2 & 33 \\ 40 & 15 \end{pmatrix}$,

3–d quartal = $\begin{pmatrix} 22.1 & 21 \\ 16.8 & 19 \\ 23 & 8 \end{pmatrix}$,

future year plan = $\begin{pmatrix} 81 & 144 \\ 70.8 & 84 \\ 102 & 37.2 \end{pmatrix}$,

output of 1–st quartal = $\begin{pmatrix} 8040 \\ 5424 \\ 3800 \end{pmatrix}$,

output of 2–nd semiannual = $\begin{pmatrix} 15092 \\ 10856 \\ 8600 \end{pmatrix}$.

9. $\begin{pmatrix} 60 & 42.75 \\ 65.8 & 56.7 \end{pmatrix}$ **10**. $\begin{pmatrix} 2.7 & 26.75 \\ 44.6 & 34.9 \end{pmatrix}$

Chapter 11

Exercises 11.1.

C. 1) *a) (11; –6), b) (9; –7), c) (–5; 8)*

2) *1^{st}.: (18; 20; 90) 2^{nd}: (20; 19; 85) 3^{rd}:(19; 22;100)*

4^{th}: (21; 18.5; 80) ***Hint***: *Det = –8*

H. 1) $A^{-1} = \frac{1}{(-1)}\begin{pmatrix} 28 & -11 & -8 \\ -49 & 19 & 14 \\ 3 & -1 & -1 \end{pmatrix}$ **2)** *1^{s}: (200; 140), 2^{nd}: (220; 120), 3^{rd}: (210; 124), 4^{th}: (200; 150).*

Exercises 11.2. **C.** 1) Doesn't exist 2) $\frac{1}{9}\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix}$

3) $\frac{1}{10}\begin{pmatrix} 14 & 2 & -4 \\ 12 & -4 & -2 \\ -16 & 2 & 6 \end{pmatrix}$ 4) Doesn't exist

H. 1) $\frac{1}{7}\begin{pmatrix} -3 & -5 \\ -7 & -14 \end{pmatrix}$ 2) Doesn't exist

3) $\frac{1}{10}\begin{pmatrix} -16 & 6 & 8 \\ 38 & -8 & -14 \\ -12 & 2 & 6 \end{pmatrix}$ 4) Doesn't exist

Summary

1. $\frac{-1}{17}\begin{pmatrix} 7 & -5 \\ -9 & 4 \end{pmatrix}$ 2. Doesn't exist

3. $\begin{pmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{pmatrix}$; 4. $\frac{1}{8}\begin{pmatrix} -3 & 6 & 1 \\ 3 & 2 & -1 \\ -1 & -6 & 3 \end{pmatrix}$;

5. $\begin{pmatrix} 1 & 1 & 5 \\ 0 & 9 & -7 \\ 1 & 6 & 1 \end{pmatrix}$; 6. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 4 \end{pmatrix}$

Exercises 11.3

1) KYRGYZSTAN; 2) SOUTH CAPITAL.

Exercises 11.4

1) NARYN SYRDARYA;

2) 42 – 39 + 36 – 29 + 46 – 38 + 50 + 31 – 43 + 46 – 38 – 50 + 31 + 48 = 32;

3) VALERY CHKALOV;

4) 45 – 33 + 46 – 37 + 48 – 43 + 35 – 46 + 33 – 45 + 30 – 38 = 46 .

Exercises 11.5

1) 454 366 402 323 510 420 446 359;

2) 416 496 388 467 378 453;

Exercises 11.6 1) GIRL; 2) CATS;
Exercises 11.7 1) MATRIX; 2) PRINCE;
Summary 2
1) –33 136 30 223 17 125 –3 72 50 235;
2) 178 22 83 84 –23 72 122 –32 108 60 –30 73.
3) GENIUS; 4) IN LOVE; 5) HEALTH; 6) I SOLVE;
7) REASON.

Chapter 12

Exercises 12.1
C. $Pf = 320 \cdot 48 + 576 \cdot 24 = 29184;$
H. $Pf = 35 \cdot 10 + 28 \cdot 12 = 686.$
Exercises 12.2
C. $C = 10 \cdot 11 + 17 \cdot 5 = 195;$ **H**. $C(4; 22) = 9.4.$
Exercises 12.3
C. $C_{min} = C(30; 70) = 1290;$ $C = x - 3y + 1470.$
H. $C_{min} = C(250; 0) = 906;$ $C = x + 1.4y + 656.$
Exercises 12.4
C. $C_{min} = C(0; 12) = 580;$ $C = 0.5x - 2y + 604.$
H. $C_{min} = C(0; 120) = 3630;$ $C = -y + 3750.$

Summary
1. 30; 135; **1a**. 70; 75; **2**. $Pf(100; 60) = 7000;$
3. $Pf(0; 78) = 12480;$ $Pf(50; 48) = 12180;$
$Pf(65; 39)=12090;$ **4**. $C(0; 130) = 390$;
5. $C(0; 14) = 280;$ $C(5; 4) = 345;$
6. C(0; 3500) = 20750; **6a**. 17000 – min;
7. 30500 – min; **7a**. 19250 – min; **8**. $C(200; 100) = 1300$
9. $C(400; 0) = 13900;$ **10**. *49; 19;*
11. **a)** $C(0; 3.5) = 7;$ **b)** $C(0; 5) = 10$**;**
c) $C(14/3; 8/3) = 58/3$**;** **12**. $z(4; 2) = 46;$
13. $z(3; 6) = z(15; 0) = 45;$ **14.** $z(4; 3) = 44$ *–min;*
$z(2; 9) = 82$ *–max;* **15**. $z(600; 400) = 5200.$

Summary for chapters 1–12

1. (400; 300; 500), (500; 200; 500), (500; 300; 400),

$$A^{-1} = \begin{pmatrix} -38 & -29 & 16 \\ -12 & -9 & 5 \\ 29 & 22 & -12 \end{pmatrix}.$$

2. $R = 111.85q$; $C = 93.75q + 452.5$; $q = 25$;

3. $Pf(60; 100) = 370$; $Pf(36; 120) = 372$;

4. $C(4; 22) = 30.8$;

5. $C_{min} = C(x; 0) = 2010$; $C = 5y + 2010$;

6. $C_{min} = C(220; 10) = 8740$; $C = 12x + 18y + 5920$;

7. PREP TO SAT $A^{-1} = \begin{pmatrix} -1 & -1 & 2 \\ -2 & -1 & 4 \\ -7 & -4 & 13 \end{pmatrix}$

8. **(**$70;\ 90;\ 80$**)**

9. a) $(436.25;\infty)$; b) $(720;\ \infty)$; c) $(400;\ 720)$.

10. a) 90^0; b) $\arccos\dfrac{-27}{\sqrt{18}\sqrt{65}}$; c) 45; d) $(8;\ 1)$;

e) $y = x - 7$; f) $y = 3x - 16$.

11. If $a = 2$, then $(p; 2 - p)$, where p is any number; if $a = 5$ there is no solution, for other values of a the system has solution $(4/(a - 5); 10/(5 - a))$.

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